

**PRACTICE SECTION 1: Basic Integration**

$$1. \int 5 \cos x dx = 5 \int \cos x dx = 5(\sin x) + C = \boxed{5\sin x + C}$$

$$2. \int 3 \csc^2 x dx = 3 \int \csc^2 x dx = 3(-\cot x) + C = \boxed{-3\cot x + C}$$

$$3. \int 18 \sec^2 x dx = 18 \int \sec^2 x dx = 18(\tan x) + C = \boxed{18\tan x + C}$$

$$4. \int -10 \sec x \tan x dx = -10 \int \sec x \tan x dx = \boxed{-10\sec x + C}$$

$$5. \int 7 \csc x \cot x dx = 7 \int \csc x \cot x dx = 7(-\csc x) + C = \boxed{-7\csc x + C}$$

$$6. \int 6x^2 dx = \frac{6x^3}{3} + C = \boxed{2x^3 + C}$$

$$7. \int 4x^5 + 3x - 8 dx = \frac{4x^6}{6} + \frac{3x^2}{2} - \frac{8x^1}{1} + C = \boxed{\frac{2}{3}x^6 + \frac{3}{2}x^2 - 8x + C}$$

$$8. \int \frac{-3}{x} dx = -3 \int \frac{1}{x} dx = \boxed{-3 \ln|x| + C}$$

$$9. \int -9e^x dx = -9 \int e^x dx = \boxed{-9e^x + C}$$

$$10. \int -2 \cdot 4^x dx = -2 \int 4^x dx = -2 \cdot \frac{1}{\ln 4} \cdot 4^x + C$$

$$= \boxed{\frac{-2}{\ln 4} \cdot 4^x + C}$$

**TOPIC C: Simplifying before integrating.**

After you clean these up, they will work as simple integration (using the antiderivatives that you totally have to memorize. Have I mentioned that those need to be MEMORIZED? Because they do. Most def. For serious.)

**PRACTICE SECTION 2:**

$$1. \int \frac{8}{x^4} dx = \int 8x^{-4} dx = \frac{8x^{-3}}{-3} + C = \boxed{\frac{8}{-3x^3} + C}$$

$$2. \int 3\sqrt{x} dx = 3 \int x^{1/2} dx = 3 \left( \frac{2}{3} x^{3/2} \right) + C = \boxed{2x^{3/2} + C}$$

↑  
got  $\frac{2}{3}$  by doing  $\frac{x^{3/2}}{3/2} = \frac{2}{3} x^{3/2}$

$$3. \int \frac{2x^2+3x-5}{x} dx = \int 2x^2 + \frac{3x}{x} - \frac{5}{x} dx = \int 2x^2 + 3 - \frac{5}{x} dx$$

OR

$$= \boxed{x^2 + 3x - 5 \ln|x| + C}$$

$$\int (2x^2 + 3x - 5)x^{-1} dx = \int 2x^2 + 3 - 5x^{-1} dx$$

$$4. \int 3x^3(5x^2 + 6x - 1) dx = \int 15x^5 + 18x^4 - 3x^3 dx = \frac{15x^6}{6} + \frac{18x^5}{5} - \frac{3x^4}{4} + C$$

$$= \boxed{\frac{5}{2}x^6 + \frac{18}{5}x^5 - \frac{3}{4}x^4 + C}$$

$$5. \int 5 \sec x (\sec x + \tan x) dx = \int 5 \sec^2 x + 5 \sec x \tan x dx$$

$$= \boxed{5 \tan x + 5 \sec x + C}$$

$$6. \int \frac{x^3+3x-5}{\sqrt{x}} dx = \int \frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} - \frac{5}{x^{1/2}} dx = \int x^{5/2} + 3x^{1/2} - 5x^{-1/2} dx$$

OR

$$\int (x^3 + 3x - 5)x^{-1/2} dx = \int x^{5/2} + 3x^{1/2} - 5x^{-1/2} dx \rightarrow$$

$$= \boxed{\frac{2}{7}x^{7/2} + 3 \cdot \frac{2}{3}x^{3/2} - 5(-2)x^{1/2} + C}$$

$$7. \int (3x+1)(x-2) dx$$

FOIL

$$= \int 3x^2 - 6x + x - 2 dx$$

$$= \int 3x^2 - 5x - 2 dx = \boxed{x^3 - \frac{5}{2}x^2 - 2x + C}$$

**TOPIC D: U-Substitution**

When you integrate a FUNCTION WITHIN A FUNCTION, you need to use u-substitution to make the complicated integral look like one of those lovely memorize antiderivatives.

**NOTE:** U-substitution is INCREDIBLY COMMON...very few integrals will ever be as simple as the straight forward antiderivatives...so you need to get comfy with using u-sub. (When you get to BC, you will learn many other techniques that you will have to use when u-sub doesn't work...but in AB, u-sub is almost always the answer.)

**HINTS FOR PICKING YOUR U:**

1. It will likely be the INSIDE function (often inside parentheses, inside a trig function, in an exponent, inside a root, or in the denominator)
2. A MULTIPLE of the "du" will generally appear elsewhere in the integrand (which is good for canceling "x stuff")

**PRACTICE SECTION 3: Basic u-substitution**

$$1. \int \sin 2x \, dx = \int \sin u \cdot \frac{du}{2} = \frac{1}{2} \int \sin u \, du = \frac{1}{2}(-\cos u) + C$$

$\begin{matrix} u=2x \\ du=2dx \\ \frac{du}{2}=dx \end{matrix}$

$$= \boxed{\frac{-1}{2} \cos 2x + C}$$

$$2. \int \cos 3x \, dx = \int \cos u \cdot \frac{du}{3} = \frac{1}{3} \int \cos u \, du = \frac{1}{3}(\sin u) + C$$

$\begin{matrix} u=3x \\ du=3dx \\ \frac{du}{3}=dx \end{matrix}$

$$= \boxed{\frac{1}{3} \sin(3x) + C}$$

$$3. \int \sec 5x \tan 5x \, dx = \int \sec u \tan u \cdot \frac{du}{5} = \frac{1}{5} \int \sec u \tan u \, du$$

$\begin{matrix} u=5x \\ du=5dx \\ \frac{du}{5}=dx \end{matrix}$

$$= \frac{1}{5} \sec u + C = \boxed{\frac{1}{5} \sec(5x) + C}$$

$$4. \int e^{x^3-6x+3} (x^2 - 2) \, dx = \int e^u \cdot \frac{(x^2-2) \, du}{3(x^2-2)} = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C$$

$\begin{matrix} u=x^3-6x+3 \\ du=(3x^2-6)dx \end{matrix}$

$$\rightarrow \frac{du}{3x^2-6}=dx \quad = \boxed{\frac{1}{3} e^{x^3-6x+3} + C}$$

$$5. \int \sqrt[3]{3x-7} \, dx = \int \sqrt[3]{u} \cdot \frac{du}{3} = \frac{1}{3} \int u^{1/3} \, du = \frac{1}{3} \left( \frac{u^{4/3}}{4/3} \right) + C$$

$\begin{matrix} u=3x-7 \\ du=3dx \\ \frac{du}{3}=dx \end{matrix}$

$$= \frac{1}{3} \cdot \frac{3}{4} u^{4/3} + C = \frac{1}{4} u^{4/3} + C$$

$$= \boxed{\frac{1}{4} (3x-7)^{4/3} + C}$$

## APAB—Basic Integration Review

because  $\frac{1}{e^u}$  is not a pattern we know.. but  $e^u$  is...

S.Hogan

$$6. \int \frac{5x}{e^{x^2}} dx = \int e^{-x^2} 5x dx = \int e^u \cdot 5x \cdot \frac{du}{-2x} = -\frac{5}{2} \int e^u du = -\frac{5}{2} e^u + C$$

$U = -x^2$   
 $du = -2x dx$   
 $\frac{du}{-2x} = dx$

$$= \boxed{-\frac{5}{2} e^{-x^2} + C}$$

$$7. \int \frac{3x-12}{x^2-8x+1} dx = \int \frac{3(x-4)}{u} \cdot \frac{du}{2(x-4)} = \frac{3}{2} \int \frac{1}{u} du = \frac{3}{2} \ln|u| + C$$

$U = x^2 - 8x + 1$   
 $du = (2x-8)dx \rightarrow \frac{du}{2x-8} = dx = \frac{du}{2(x-4)}$

$$= \boxed{\frac{3}{2} \ln|x^2 - 8x + 1| + C}$$

$$8. \int x^2 \sin(x^3) dx = \int x^2 \sin u \cdot \frac{du}{3x^2} = \frac{1}{3} \int \sin u du = \frac{1}{3} (-\cos u) + C$$

$U = x^3$   
 $du = 3x^2 dx$   
 $\frac{du}{3x^2} = dx$

$$= \boxed{-\frac{1}{3} \cos(u) + C} = \boxed{-\frac{1}{3} \cos(x^3) + C}$$

$$9. \int \sin^4 x \cos x dx = \int u^4 \cos x \cdot \frac{du}{\cos x} = \int u^4 du = \frac{u^5}{5} + C$$

$U = \sin x$   
 $du = \cos x dx$   
 $\frac{du}{\cos x} = dx$

$$= \boxed{\frac{\sin^5 x}{5} + C}$$

$$10. \int \cot^5 x \csc^2 x dx = \int u^5 \csc^2 x \cdot \frac{du}{-\csc^2 x} = - \int u^5 du = -\frac{u^6}{6} + C$$

$U = \cot x$   
 $du = -\csc^2 x dx$   
 $\frac{du}{-\csc^2 x} = dx$

$$= \boxed{-\frac{\cot^6 x}{6} + C}$$

$$11. \int 2^{3x} dx = \int 2^u \frac{du}{3} = \frac{1}{3} \int 2^u du = \frac{1}{3} \frac{1}{\ln 2} \cdot 2^u + C$$

$U = 3x$   
 $du = 3dx$   
 $\frac{du}{3} = dx$

$$= \boxed{\frac{1}{3 \ln 2} 2^{3x} + C}$$

$$12. \int 3x(4x^2 - 5)^6 dx = \int 3x(u)^6 \cdot \frac{du}{8x} = \frac{3}{8} \int u^6 du = \frac{3}{8} \frac{u^7}{7} + C$$

$U = 4x^2 - 5$   
 $du = 8x dx$   
 $\frac{du}{8x} = dx$

$$= \boxed{\frac{3}{56} (4x^2 - 5)^7 + C}$$

$$13. \int 4\tan x \, dx = 4 \int \frac{\sin x}{\cos x} \, dx = 4 \int \frac{\sin x}{u} \cdot \frac{du}{-\sin x}$$

$U = \cos x$   
 $du = -\sin x \, dx$   
 $\frac{du}{-\sin x} = dx$

$$= -4 \int \frac{1}{u} \, du = -4 \ln|u| + C$$

$$= -4 \ln|\cos x| + C$$

$$14. \int \frac{5}{(2x-3)^2} \, dx = \int \frac{5}{u^2} \cdot \frac{du}{2} = \int \frac{5}{2} u^{-2} \, du = \frac{\frac{5}{2} u^{-1}}{-1} + C$$

$U = 2x-3$   
 $du = 2 \, dx$   
 $\frac{du}{2} = dx$

$$= -\frac{5}{2u} + C = -\frac{5}{2(2x-3)} + C$$

Sometimes, when you use u-substitution, you are fairly certain you've picked the correct "U" but some of the "x stuff" doesn't cancel. Since we can't integrate when there are multiple variables, we need to find a separate way (using our equation for "U") to replace the remaining "x stuff."

## PRACTICE SECTION 4: Slightly more complicated u-substitution

$$(1) \int 2x\sqrt{x-3} \, dx = \int 2x\sqrt{u} \, du = \int 2(u+3)\sqrt{u} \, du = \int 2u^{1/2}(u+3) \, du$$

$U = x-3$   
 $du = dx$

replace  $u/u$   
 $U = x-3$   
 $U+3 = x$

$$= \int 2u^{3/2} + 6u^{1/2} \, du$$

$$= 2u^{5/2} + 6u^{3/2} + C$$

$$\left( \frac{2}{5}u^{5/2} + 6\left(\frac{2}{3}u^{3/2}\right) + C \right)$$

$$= 2\left(\frac{2}{5}\right)u^{5/2} + 6\left(\frac{2}{3}\right)u^{3/2} + C = \left[\frac{4}{5}(x-3)^{5/2} + 4(x-3)^{3/2} + C\right]$$


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$$(2) \int \frac{3x-1}{(x+1)^2} \, dx = \int \frac{3x-1}{u^2} \, du = \int \frac{3(u-1)-1}{u^2} \, du = \int \frac{3u-4}{u^2} \, du = \int \frac{3u-4}{u^2} \, du$$

$U = x+1$   
 $du = dx$

replace  
 $U = x+1$   
 $U-1 = x$

$$= \int \frac{3u}{u^2} - \frac{4}{u^2} \, du = \int \frac{3}{u} - 4u^{-2} \, du = 3\ln|u| - \frac{4u^{-1}}{-1} + C = 3\ln|u| + \frac{4}{u} + C$$

$$= [3\ln|x+1| + \frac{4}{x+1} + C]$$

*SKILL*

$$(3) \int x^2(x^2-4)^8 \, dx = \int x^2 u^8 \cdot \frac{du}{2x} = \frac{1}{2} \int x \cdot u^8 \, du = \frac{1}{2} \int u^8 \sqrt{u+4} \, du$$

$U = x^2-4$   
 $du = 2x \, dx$   
 $\frac{du}{2x} = dx$

$U = x^2-4$   
 $U+4 = x^2$   
 $\pm \sqrt{u+4} = x$

\* OOPS... I meant  
to make this  
 $\int x^3(x^2-4)^8 \, dx$   
cubed.

OOPS... WRITE/TYPO  
You CAN'T integrate  
THIS one now  
IT ALL OUT  
SORRY!!

## Indefinite Integrals

$$dx = \int f(x)dx \pm \int g(x)dx$$

$$\int f(x)dx$$

ves

these antiderivatives. There is no wiggle room here. Make  
f, whatever it takes...but you need to have these in you

PS4 #3 (the one with the TYPO)

$$\int x^3 (x^2 - 4)^8 dx = \int x^3 (u)^8 \frac{du}{dx}$$

$u = x^2 - 4$   
 $du = 2x dx$   
 $\frac{du}{dx} = dx$

can't have an integral w/ mixed variables

$$= \int \frac{1}{2} x^2 u^8 du$$

↑  
 need to replace  $x^2$   
 with  $u$  stuff  
 $u = x^2 - 4$   
 $u + 4 = x^2$

$$= \frac{1}{2} \int (u+4) u^8 du$$

$$= \frac{1}{2} \int u^9 + 4u^8 du$$

$$= \frac{1}{2} \left( \frac{u^{10}}{10} + \frac{4u^9}{9} \right) + C$$

$$= \frac{u^{10}}{20} + \frac{2u^9}{9} + C$$

$$= \boxed{\frac{(x^2 - 4)^{10}}{20} + 2 \frac{(x^2 - 4)^9}{9} + C}$$

$$\ln|u| + C$$

$$C$$

$$+ C$$

sines...you can use the table to help you derive (move  
move UP) sines and cosines.

WE	$\sin(u)$ $\cos(u)$ $-\sin(u)$ $-\cos(u)$	$\uparrow$ $\uparrow$ $\uparrow$ <b>INTEGRATE</b>
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**PART II: DEFINITE INTEGRALS (with bounds)****TOPIC E: Properties of definite integrals on  $[a, b]$  with  $a \leq c \leq b$ .**

1.  $\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
2.  $\int_a^b k f(x) dx = k \int_a^b f(x) dx$
3.  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
4.  $\int_b^a f(x) dx = - \int_a^b f(x) dx$
5.  $\int_a^a f(x) dx = 0$
6.  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$  for EVEN FXN  $f(x)$ .
7.  $\int_{-a}^a f(x) dx = 0$  for ODD FXN  $f(x)$ .

**TOPIC F: The First Fundamental Theorem of Calculus (FTC).**Let  $F(x)$  be the antiderivative of  $f(x)$  (here the capitalization matters...FYI.)

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

Meaning that you:

- (1) integrate the function (after you integrate, you stop writing the " $\int dx$ " as those are telling you to integrate)
- (2) write the antiderivative in brackets with the bounds as super and sub scripts on the right side
- (3) evaluate the antiderivative at the top bound and subtract the value of the antiderivative evaluated at the bottom bound.

This is ALWAYS how we evaluate a definite integral.

NOTE: Your calculator can evaluate DEFINITE integrals. You do this by using MATH 9 to get "FNINT(\*...the calculator syntax is as follows:

$$\int_a^b f(x) dx = FNINT(f(x), x, a, b)$$

If the function is really complicated, it might be worth putting the function into the  $Y_1$  spot and then using  $FNINT(Y_1, X, A, B)$ **PRACTICE SECTION 5: The 1<sup>st</sup> FTC—Evaluate each integral**

(by hand AND on your calculator to check your work)

$$(1) \int_1^3 x^2 dx = \left[ \frac{x^3}{3} \right]_1^3 = \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}$$

$$(2) \int_0^{\pi/4} \sin(3x) dx = \left[ -\frac{1}{3} \cos(3x) \right]_0^{\pi/4} = -\frac{1}{3} \cos\left(\frac{3\pi}{4}\right) + \frac{1}{3} \cos(0)$$

$$\begin{aligned} & \int_0^{\pi/4} \sin(3x) dx \\ & \int_0^{\pi/4} \sin u \cdot \frac{du}{3} = \frac{1}{3} \int_0^{\pi/4} \sin u du \\ & \frac{1}{3} \left[ -\cos u \right]_0^{\pi/4} = \frac{1}{3} \left[ -\cos\left(\frac{\pi}{4}\right) + \cos(0) \right] \\ & \frac{1}{3} \left[ -\cos\left(\frac{\pi}{4}\right) + \cos(0) \right] = \frac{1}{3} \left[ -\frac{\sqrt{2}}{2} + 1 \right] = \boxed{\frac{\sqrt{2}}{6} + \frac{1}{3}} \end{aligned}$$

$$\begin{aligned}
 u &= 2x \\
 du &= 2dx \\
 \frac{du}{2} &= dx \\
 \int e^{2x} dx &= \int e^u \frac{du}{2} = \frac{1}{2} \int e^u du = \frac{1}{2} [e^u] \\
 (3) \int_0^4 e^{2x} dx &= \frac{1}{2} [e^{2x}]_0^4 = \frac{1}{2} (e^8 - e^0) = \boxed{\frac{1}{2}(e^8 - 1)} \\
 &\text{OR} \\
 &\boxed{\frac{e^8 - 1}{2}}
 \end{aligned}$$
  

$$(4) \int_1^3 \frac{1}{x} dx = [\ln|x|]_1^3 = \ln 3 - \ln 1 = \boxed{\ln 3}$$

$$\begin{aligned}
 (5) \int_0^2 2x^3 - 4x + 3 dx &= \left[ \frac{2}{4}x^4 - \frac{4}{2}x^2 + 3x \right]_0^2 = \left[ \frac{1}{2}x^4 - 2x^2 + 3x \right]_0^2 \\
 \left( \frac{1}{2}(16) - 2(4) + 3(2) \right) - (0+0+0) &= 8 - 8 + 6 = \boxed{6}
 \end{aligned}$$

$$\begin{aligned}
 u &= 2x+1 \\
 du &= 2dx \\
 dx &= \frac{1}{2}du \\
 \int u^{1/2} \frac{du}{2} &= \frac{1}{2} \int u^{1/2} du = \boxed{\frac{1}{2} \cdot \frac{2}{3} u^{3/2}} \\
 (6) \int_0^4 \sqrt{2x+1} dx &= \frac{1}{3} [(2x+1)^{3/2}]_0^4 = \frac{1}{3} (8+1)^{3/2} - \frac{1}{3} (0+1)^{3/2} \\
 &= \frac{1}{3} (9)^{3/2} - \frac{1}{3} (1)^{3/2} = \frac{1}{3} (27) - \frac{1}{3} = \boxed{8}
 \end{aligned}$$

$$\begin{aligned}
 \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \quad (7) \int_{\pi/4}^{\pi/3} \sec^2 x dx &= [\tan x]_{\pi/4}^{\pi/3} = \tan \frac{\pi}{3} - \tan \frac{\pi}{4} = \frac{\sqrt{3}/2}{1/2} - \frac{\sqrt{2}/2}{\sqrt{2}/2} \\
 \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) &= \boxed{\sqrt{3} - 1}
 \end{aligned}$$

**TOPIC G: The Mean Value Theorem (MVT) for Integrals**For a continuous function on  $[a, b]$  with  $a \leq c \leq b$ .

$$\int_a^b f(x) dx = (b-a)f(c)$$

For at least one  $c$ ,  $f(c)$  is called the "Average value of  $f(x)$  on  $[a, b]$ "**TOPIC H: The Average Value of a Function**

$$\frac{1}{(b-a)} \int_a^b f(x) dx = f(c)$$

I always remember this as the "integral divided by the width of the interval." This makes sense if you picture a rectangle...the area of a rectangle is width times height ( $A = W \cdot H$ )

If you solve for height, you get  $H = A/W$ . Thus, the average height ( $y$ -value) would be the AREA (a.k.a. integral) divided by the WIDTH (a.k.a. "b minus a".)

**PRACTICE SECTION 6: Find the average value of a function on a given interval.**

(1)  $f(x) = 3x^2$  on  $[1, 4]$

$$\text{Avg} = \frac{1}{4-1} \int_1^4 3x^2 dx = \frac{1}{3} [x^3]_1^4$$

$$= \frac{1}{3} (4^3 - 1^3) = \frac{1}{3} (64 - 1) = \frac{1}{3} (63) = \boxed{21}$$

(2)  $f(x) = x^2 - 4x$  on  $[0, 5]$

$$\text{Avg} = \frac{1}{5-0} \int_0^5 x^2 - 4x dx = \frac{1}{5} \left[ \frac{x^3}{3} - 2x^2 \right]_0^5$$

$$= \frac{1}{5} \left( \frac{125}{3} - 2(25) \right) - \left( \frac{1}{5}(0-0) \right) = \frac{1}{5} \left( \frac{125}{3} - 50 \right) = \frac{1}{5} \left( \frac{125}{3} - \frac{150}{3} \right) = \frac{-25}{5(3)} = \boxed{\frac{-5}{3}}$$

(3)  $f(x) = \frac{2}{x^2}$  on  $[1, 3]$

$$\text{Avg} = \frac{1}{3-1} \int_1^3 \frac{2}{x^2} dx = \frac{1}{2} \int 2x^{-2} dx = \frac{1}{2} \left[ \frac{2x^{-1}}{-1} \right]_1^3 = \frac{1}{2} \left[ \frac{-2}{x} \right]_1^3$$

$$= \frac{1}{2} \left( -\frac{2}{3} \right) - \frac{1}{2} \left( -\frac{2}{1} \right) = -\frac{2}{6} + \frac{2}{2} = -\frac{1}{3} + \frac{1}{1} = \boxed{\cancel{-\frac{1}{3}}} \quad \frac{-1+3}{3} = \boxed{\frac{2}{3}}$$

(4)  $f(x) = \frac{5}{x}$  on  $[1, e]$

$$\text{Avg} = \frac{1}{e-1} \int_1^e \frac{5}{x} dx = \frac{1}{e-1} [5 \ln|x|]_1^e = \frac{1}{e-1} [5 \ln e - 5 \ln 1] = \boxed{\frac{5 \ln e}{e-1}}$$

(5)  $f(x) = \sin x$  on  $[0, \pi/3]$

$$\text{Avg} = \frac{1}{\frac{\pi}{3}-0} \int_0^{\pi/3} \sin x dx = \frac{1}{(\pi/3)} [-\cos x]_0^{\pi/3} = \frac{3}{\pi} \left( -\cos \frac{\pi}{3} - (-\cos 0) \right)$$

$$= \frac{3}{\pi} \left( -\frac{1}{2} + 1 \right) = \frac{3}{\pi} \left( \frac{1}{2} \right) = \boxed{\frac{3}{2\pi}}$$

(6)  $f(x) = e^{3x}$  on  $[1, 5]$

$$\begin{aligned} \text{Avg} &= \frac{1}{5-1} \int_1^5 e^{3x} dx = \frac{1}{4} \int_1^5 e^{3x} dx \\ &= \frac{1}{4} \left[ \frac{1}{3} e^{3x} \right]_1^5 = \frac{1}{12} \left[ e^{3x} \right]_1^5 \\ &= \frac{1}{12} (e^{15} - e^3) = \boxed{\frac{e^{15} - e^3}{12}} \end{aligned}$$

$$\left\{ \begin{array}{l} \int e^{3x} dx = \int e^u \frac{du}{3} \\ u = 3x \\ du = 3dx \\ \frac{du}{3} = dx \\ = \frac{1}{3} \int e^u du = \frac{1}{3} [e^u] \\ = \left[ \frac{1}{3} e^{3x} \right] \end{array} \right.$$

**TOPIC I: The Second Fundamental Theorem of Calculus (FTC)**

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$

NOTE: Most of the time, when you see this, one of the bounds is a constant, so it's derivative is zero. The keys to using the 2<sup>nd</sup> FTC are:

- (1) the derivative and the integral "cancel"
- (2) the bounds get plugged into the function that is in the integrand
- (3) because of the chain rule, you have to derive the bounds after you plug them in

**PRACTICE SECTION 7: 2<sup>nd</sup> FTC**

Use the 2<sup>nd</sup> FTC to evaluate each derivative.

$$(1) \frac{d}{dx} \int_6^{5x} \sin t dt = \sin(5x) \cdot 5 - \sin(6) \cdot 0 \\ = \boxed{5\sin(5x)}$$

$$(2) \frac{d}{dx} \int_3^{x^3} e^t dt = e^{x^3} \cdot 3x^2 - e^3 \cdot 0 = \boxed{3x^2 e^{x^3}}$$

$$(3) \frac{d}{dx} \int_{\sin x}^{10} \sqrt{2t+3} dt = \sqrt{20+3}(0) - \sqrt{2\sin x+3} \cdot \cos x \\ = \boxed{-\cos x \sqrt{2\sin x+3}}$$

$$(4) \frac{d}{dx} \int_{6x}^{3x^2} \ln t dt = \ln(3x^2) \cdot 6x - \ln(6x) \cdot 6 \\ = \boxed{6x \ln(3x^2) - 6 \ln 6x}$$

**TOPIC J: Integrals that are best evaluated by geometry**

Because  $\int_a^b f(x) dx$  means the AREA under the curve from [a,b], sometimes we can use simple geometric formulae as a faster, more efficient way to integrate. This is most common for:

- (1) Linear absolute values (which graph like a V...so the area ends up being two triangles.) To figure out where the "point" of the V happens, set in inside of the linear absolute value equal to zero and solve for x. Sometimes, depending on the interval, you may also have a piece that is a trapezoid rather than a triangle.
- (2) Semicircles, quarter circles, and circles (which generally have the equation  $y = \pm\sqrt{r^2 - x^2}$  with radius = r, when centered at the origin)
  - a. So  $y = \sqrt{r^2 - x^2}$  from [-r, r] is a semicircle of radius r. (the TOP half of the circle)
  - b. So  $y = -\sqrt{r^2 - x^2}$  from [-r, r] is a semicircle of radius r. (the BOTTOM half of the circle)
  - c. Either of the two listed above would be a QUARTER circle if the interval was [0, r] or [-r, 0]

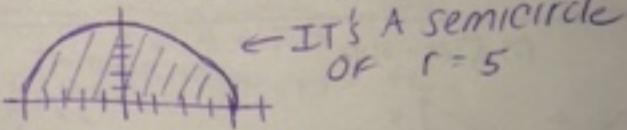
IN ALL OF THESE CASES, SKETCH A QUICK GRAPH....IT IS TOTALLY WORTH IT IN THESE SCENARIOS...I PROMISE.

NOTE: The area of a triangle is  $A = \frac{1}{2}bh$ , the area of a circle is  $A = \pi r^2$ , and the area of a trapezoid is  $A = \frac{1}{2}h(b_1 + b_2)$ ...these are obviously three formulae that you should have memorized from geometry.

**PRACTICE SECTION 8: Evaluate these integrals by sketching the graph and using geometry.**

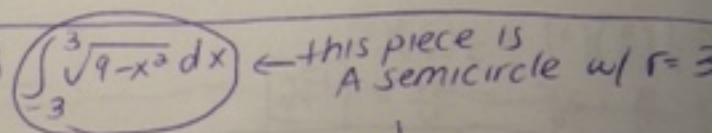
$$(1) \int_{-5}^5 \sqrt{25 - x^2} dx$$

$$\begin{aligned} A &= \frac{1}{2}\pi(5)^2 \\ &= \boxed{\frac{25}{2}\pi} = \int_{-5}^5 \sqrt{25-x^2} dx \end{aligned}$$



$$(2) \int_{-3}^3 2\sqrt{9 - x^2} dx$$

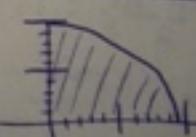
$$\begin{aligned} &= 2 \int_{-3}^3 \sqrt{9-x^2} dx \\ &= 2 \left[ \frac{1}{2}\pi(3)^2 \right] = \boxed{9\pi} \end{aligned}$$



$$(3) \int_0^{10} \sqrt{100 - x^2} dx$$

$$= \frac{1}{4}\pi(10)^2 = \boxed{25\pi}$$

← IT'S  
A QUARTER  
CIRCLE OF r = 10



APAB—Basic Integration Review

coefficient  
this is  
a quarter  
circle w/ r=6 !! S.Hogan

$$(4) \int_{-6}^0 -3\sqrt{36-x^2} dx = -3 \int_{-6}^0 \sqrt{36-x^2} dx$$

$$= -3 \left( \frac{1}{4} \pi (6)^2 \right) = -3 \left( \frac{36}{4} \pi \right) = \boxed{-27\pi}$$

(5)  $\int_0^5 |x-3| dx$

it's a "V" with its point at  $x=3$   
and slope of  $\pm 1$   
~~(m=1, x>3)~~  
~~(m=-1, x<3)~~

$$= 3 \triangle + \triangle^2$$

$$= \frac{1}{2}(3)(3) + \frac{1}{2}(2)(2) = \frac{9}{2} + \frac{4}{2} = \boxed{\frac{13}{2}}$$

(6)  $\int_0^6 |2x-4| dx$

it's a "V" with its point at  $2x-4=0$   
 $x=2$   
and slope of  $\pm 2$   
 $(m=2, x>2)$   
 $(m=-2, x<2)$

$$= 4 \triangle + \triangle^8$$

$$= \frac{1}{2}(2)(4) + \frac{1}{2}(4)(8)$$

$$= 4 + 16 = \boxed{20}$$

(7)  $\int_1^4 |2x-3| dx$

it's a "V" with point at  $2x-3=0$   
 $x=\frac{3}{2}$   
and slope of  $\pm 2$   
 $(m=2, x>\frac{3}{2})$   
 $(m=-2, x<\frac{3}{2})$

$$= 3 \triangle + \triangle^5$$

$$= \frac{1}{2}\left(\frac{3}{2}\right)(3) + \frac{1}{2}\left(\frac{5}{2}\right)(5)$$

$$= \frac{9}{4} + \frac{25}{4} = \boxed{\frac{17}{2}}$$

(8)  $\int_1^3 |2x+1| dx$

it's a "V" with point at  $2x+1=0$   
 $x=-\frac{1}{2}$   
and slope  
 $m=\pm 2$   
 $(m=2, x>-\frac{1}{2})$   
 $(m=-2, x<-\frac{1}{2})$

$$= 3 \triangle^7$$

$$= \frac{1}{2}(2)(3+7)$$

$$= 1(10) = \boxed{10}$$

so it's a TRAPEZOID

### PART III: APPLICATIONS OF INTEGRALS

#### TOPIC K: Solving differential equations (Diff EQs)

When you solve diff EQs, you will be integrating WITHOUT BOUNDS. This means that every time you integrate, you will need to write a  $+C$ . Then you will use a give point (often called an "initial value") to find the value of  $C$ .

If you have to integrate multiple times, you need to find the " $C$ " at every step before you do the next round of integrating.

- (1) YOU MUST WRITE BOTH THE INTEGRAL SYMBOL AND THE  $d\_\_$  (insert correct variable of integration here.) You cannot have an integral symbol without having the " $dx$ " (though it is not always an  $x$ .) If the variable of integration is missing, the integral will immediately lose credit on the AP.
- (2) YOU MUST WRITE WHICH FUNCTION YOU HAVE IN EACH STEP ( $f(x)$ ,  $f'(x)$ ,  $f''(x)$ )...and you can only say things are equal if there are actually equal.  
WRITE WITH PURPOSE.

#### PRACTICE SECTION 9: Solving Diff EQs

Given the information, write the equation for  $f(x)$ .

$$(1) f'(x) = 2x, f(1) = 10$$

$$f(x) = \int f'(x) dx = \int 2x dx = x^2 + C$$

$$f(1) = 1^2 + C = 10 \\ C = 9$$

$$\boxed{f(x) = x^2 + 9}$$

$$(2) f'(x) = \sin x, f\left(\frac{\pi}{2}\right) = 2$$

$$f(x) = \int f'(x) dx = \int \sin x dx = -\cos x + C$$

$$f\left(\frac{\pi}{2}\right) = 2 = -\cos \frac{\pi}{2} + C = 0 + C \rightarrow C = 2$$

$$\boxed{f(x) = -\cos x + 2}$$

$$(3) f'(x) = e^{3x}, f(0) = -4$$

$$f(x) = \int f'(x) dx = \int e^{3x} dx = \int e^u \frac{du}{3} = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C$$

$$f(0) = -4 = \frac{1}{3} e^0 + C$$

$$-4 = \frac{1}{3} + C$$

$$-4 - \frac{1}{3} = -\frac{12}{3} - \frac{1}{3} = -\frac{13}{3}$$

$$C = -\frac{13}{3}$$

$$\boxed{f(x) = \frac{1}{3} e^{3x} - \frac{13}{3}}$$

$$(4) f''(x) = 6x, f'(-1) = 5, f(0) = 3$$

$$f'(x) = \int f''(x) dx = \int 6x dx = 3x^2 + C$$

$$\underline{f'(-1) = 5 = 3(-1)^2 + C} = 3 + C \rightarrow C = 2$$

$$\boxed{f'(x) = 3x^2 + 2}$$

$$f(x) = \int f'(x) dx = \int 3x^2 + 2 dx = x^3 + 2x + C,$$

$$f(0) = 3 = 0 + 0 + C, \rightarrow C_1 = 3$$

$$\boxed{f(x) = x^3 + 2x + 3}$$

$$(5) f''(x) = 4x + 5, f'(2) = 5, f(0) = 2$$

$$f'(x) = \int f''(x) dx = \int 4x + 5 dx = 2x^2 + 5x + C$$

$$\begin{aligned} f'(2) &= 5 = 2(4) + 5(2) + C \\ 5 &= 8 + 10 + C \\ -13 &= C \end{aligned} \rightarrow \boxed{f'(x) = 2x^2 + 5x - 13}$$

$$f(x) = \int f'(x) dx = \int 2x^2 + 5x - 13 dx = \frac{2}{3}x^3 + \frac{5}{2}x^2 - 13x + C,$$

$$f(0) = 2 = 0 + 0 + 0 + C, \rightarrow C_1 = 2$$

$$\boxed{f(x) = \frac{2}{3}x^3 + \frac{5}{2}x^2 - 13x + 2}$$

$$(6) f''(x) = \sqrt{x}, f'(4) = 0, f(1) = 5$$

$$f'(x) = \int f''(x) dx = \int \sqrt{x} dx = \int x^{1/2} dx = \frac{2}{3}x^{3/2} + C$$

$$f'(4) = 0 = \frac{2}{3}(4)^{3/2} + C$$

$$0 = \frac{2}{3}(8) + C$$

$$-\frac{16}{3} = C$$

$$f(x) = \int f'(x) dx = \int \frac{2}{3}x^{3/2} - \frac{16}{3} dx = \frac{2}{3} \cdot \frac{2}{5}x^{5/2} - \frac{16}{3}x + C_1$$

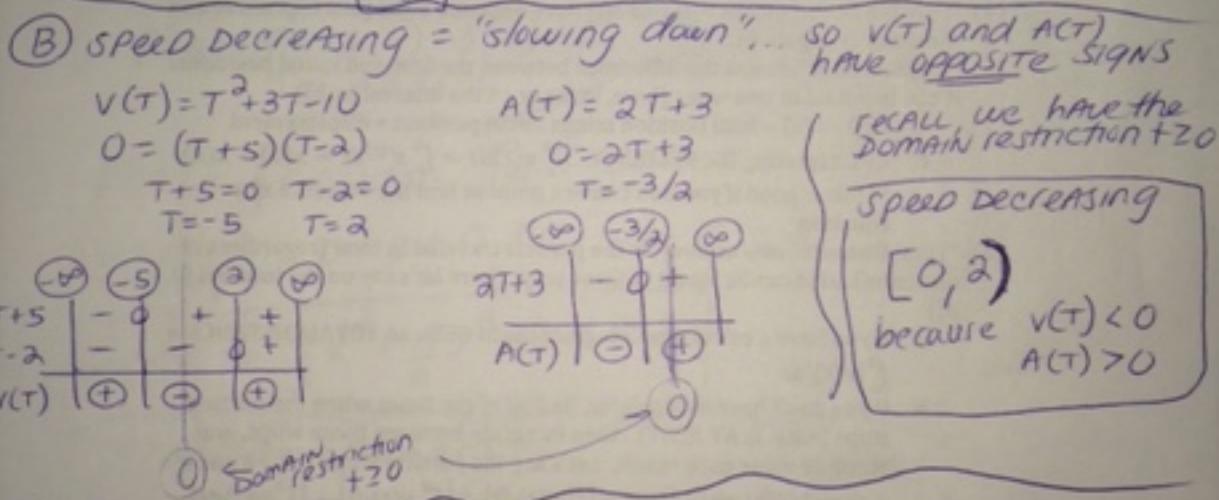
$$f(x) = \frac{4}{15}x^{5/2} - \frac{16}{3}x + C_1 \rightarrow \frac{75}{15} = \frac{4}{15} - \frac{80}{15} + C_1$$

$$\begin{aligned} f(1) &= 5 = \frac{4}{15} - \frac{16}{3} + C_1 \\ \frac{75}{15} + \frac{76}{15} &= C_1 \\ \frac{151}{15} &= C_1 \end{aligned} \rightarrow \boxed{f(x) = \frac{4}{15}x^{5/2} - \frac{16}{3}x + \frac{151}{15}}$$

PRACTICE SECTION 10: AVP Problems

1. (Non-Calculator) The velocity of a moving particle on a coordinate line is given by  $v(t) = t^2 + 3t - 10$  ft/min, where  $t$  is measured in minutes and  $t \geq 0$ .
- Find the displacement of the particle during the first 3 minutes.
  - Find when the particle's speed is decreasing. Justify.
  - Write, but do not evaluate, an integral expression to find the total distance traveled by the particle for the first 5 minutes.

$$\begin{aligned} A) \int_0^3 v(t) dt &= \int_0^3 t^2 + 3t - 10 dt = \left[ \frac{t^3}{3} + \frac{3t^2}{2} - 10t \right]_0^3 \\ &= \left( \frac{27}{3} - \frac{27}{2} - 30 \right) - (0 - 0 - 0) = 9 - \frac{27}{2} - 30 = -\frac{27}{2} - 21 \\ &= -\frac{27}{2} - \frac{42}{2} = \boxed{-\frac{69}{2}} \end{aligned}$$



C)  $\int_0^5 |v(t)| dt = \boxed{\int_0^5 |t^2 + 3t - 10| dt}$

TOTAL DISTANCE is the integral of speed.

OR

$$-\int_0^2 t^2 + 3t - 10 dt + \int_2^5 t^2 + 3t - 10 dt$$

because  $v(t) < 0$  so travel  $\downarrow$

2. (Non-Calculator) Given the acceleration of the particle is  $a(t) = -4 \text{ ft/sec}^2$  and  $v(0) = 12 \text{ ft/sec}$  during the interval  $0 \leq t \leq 8$ .
- Find the average velocity of the particle for the interval  $0 \leq t \leq 8$ .
  - Find when the instantaneous velocity of the particle is equal to the average velocity from part (a).
  - Find when the velocity is increasing.
  - Find the total distance traveled by the particle during the interval  $0 \leq t \leq 8$ .

good starting place from  $a(t)$  is to integrate to find velocity.

(A)  $v(t) = \int a(t) dt = \int -4 dt = -4t + C$   
 $v(t) = -4t + C \quad v(0) = 12 = 0 + C \rightarrow C = 12$

$$\begin{aligned} v(t) &= -4t + 12 \\ \text{Avg } v(t) &= \frac{1}{8-0} \int_0^8 v(t) dt = \frac{1}{8} \int_0^8 -4t + 12 dt = \frac{1}{8} \left[ -2t^2 + 12t \right]_0^8 \\ &= \frac{1}{8} \left[ -2(64) + 12(8) - (0+0) \right] = \frac{-2(64)}{8} + \frac{12(8)}{8} = -16 + 12 \\ &= -4 \text{ ft/sec} \end{aligned}$$

(B) instant velocity is  $v(t)$   
 $-4t + 12 = -4$   
 $-4t = -16$   
 $t = 4 \text{ sec}$

(C)  $v(t)$  velocity is increasing when  $v'(t)$  is  $\oplus$  so when  $a(t) > 0$   
 $a(t) = -4$  so  $a(t) > 0$  never thus velocity is never increasing

(D) b/c it's a non-calculator section  $\int v(t) dt$  is not a great plan...  
so instead...

① FIND when particle is at rest (stopped)  $v(t) = -4t + 12 = 0$   
 $-4t = -12$   
 $t = 3$

② integrate between stops and make each displacement  $\oplus$   $\left| \int_0^3 -4t + 12 dt \right| + \left| \int_3^8 -4t + 12 dt \right|$

$$\begin{aligned} &= \left| \left[ -2t^2 + 12t \right]_0^3 \right| + \left| \left[ -2t^2 + 12t \right]_3^8 \right| = \left| (-18 + 36) - (0+0) \right| + \left| (-128 + 96) - (-18 + 36) \right| \\ &= |18| + |-32 - 18| = |18| + |-50| = 18 + 50 = 68 \text{ feet} \leftarrow \text{TOTAL DISTANCE} \end{aligned}$$

## APAB—Basic Integration Review

*ACTUALLY solvable by hand FYI, but if you're permitted to use a calculator S. Hogan ACTUALLY DO SO*

3. (Calculator) The velocity function of a moving particle is  $v(t) = 3\cos(2t)$  in/hr for  $0 \leq t \leq 2\pi$  hours.

- Determine when the particle is moving to the right. Justify.
- Find the total distance traveled by the particle during the time interval  $0 \leq t \leq 2\pi$  hours.
- Given  $x(0) = 5$ , find  $x(6)$ .
- Find when the particle is speeding up.

(A) Particle is moving to the right when  $v(t) \geq 0$

$$Y_1 = 3\cos(2x)$$

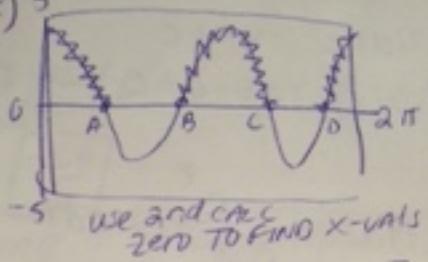
Graph in window

$$X_{\min} = 0$$

$$X_{\max} = 2\pi$$

$$Y_{\min} = -5$$

$$Y_{\max} = 5$$



DO NOT round when using them  
(converting)  
ANS

$$A: x \approx 7.8539816 = a$$

$$B: x \approx 2.3561945 = b$$

$$C: x \approx 3.9269908 = c$$

$$D: x \approx 5.4977871 = d$$

ANS:

$$\text{moving right: } [0, a) \cup (b, c) \cup (d, 2\pi]$$

$$\text{or } [0, 7.85) \cup (2.356, 3.926) \cup (5.497, 2\pi]$$

(B)  $\text{TOTAL DIST} = \int_0^{2\pi} |v(t)| dt = \text{FNINT}(\text{abs}(Y_1), X, 0, 2\pi) \approx 12$

*must have this for AP credit!*

*NO AP credit for this*

(C)  $x(0) = 5$  FIND  $x(6)$   
we know from 1st FTC:  $\int_0^6 v(t) dt = x(6) - x(0)$

*can be found w/ calculator*

*given*

thus  $x(6) = x(0) + \int_0^6 v(t) dt = 5 + \text{FNINT}(Y_1, X, 0, 6) \approx 4.195$

*must have this AP credit*

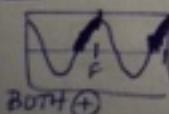
*NO AP credit for calculator syntax*

(D) SPEEDING UP MEANS  $v(t)$  AND  $a(t)$  HAVE THE SAME SIGN

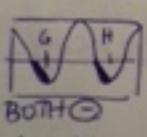
either  $v(t) > 0$ ,  $a(t) > 0$  which looks like  $v(t)$  is above x-axis and

OR  $v(t) < 0$ ,  $a(t) < 0$  which looks like  $v(t)$  is below x-axis and

$a(t) = \text{slope of } v(t) \text{ is positive}$



use 2nd calc to find  
max F  
 $x \approx 3.11592$



use 2nd calc to find x=G, H  
 $x \approx 1.570296, 4.7133882$

ANS:  $(2.356, 3.14)$  U  $(5.497, 2\pi)$  U  $(7.85, 1.57)$   
 $\cup (3.926, 4.71)$

4. (Calculator) A particle moves along the  $x$ -axis so that its velocity at any time  $t \geq 0$  is given by  $v(t) = 3t^2 - 2t - 1$ . The position  $x(t)$  is 5 for  $t = 2$ .

a) Write a polynomial expression for the position of the particle at any time  $t \geq 0$ .

b) For what values of  $t$ ,  $0 \leq t \leq 3$ , is the particle's instantaneous velocity the same as its

average velocity on the closed interval  $[0, 3]$ ?

c) Find the total distance traveled by the particle from time  $t = 0$  until time  $t = 3$ .

$$\textcircled{A} \quad x(t) = \int v(t) dt = \int 3t^2 - 2t - 1 dt = t^3 - t^2 - t + C$$

$$x(2) = 5 = 2^3 - 2^2 - 2 + C = 8 - 4 - 2 + C = 2 + C$$

$$5 = 2 + C \rightarrow C = 3$$

*your calculator  
CANNOT do this part*

$$\boxed{x(t) = t^3 - t^2 - t + 3}$$

\textcircled{B} INSTANT velocity

$$v(t) = 3t^2 - 2t - 1$$

$$\text{Avg. velocity}$$

$$x(3) - x(0) = \frac{(87 - 9 - 3 + 3) - (0 + 0 + 0 + 3)}{3 - 0}$$

$$= \frac{18 - 3}{3} = \frac{15}{3} = 5$$

*( $3t^2 - 2t - 1 = 5$ ) use your calculator to solve*

$$\underbrace{3t^2 - 2t - 6}_Y = 0$$

*2nd zero*

$$T = -1.119603 \leftarrow \text{NOT IN DOMAIN } T \geq 0$$

$$T = 1.7862996$$

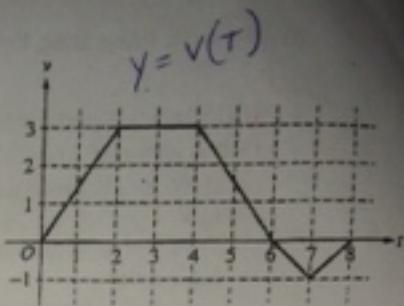
ANS:  $T = 1.786$

$$\textcircled{C} \quad \text{TOTAL DIST} = \int_0^3 |v(t)| dt = \text{FNINT}(\text{abs}(3x^2 - 2x - 1), x, 0, 3) \approx \boxed{17.000}$$

*MUST have  
this for  
AP CREDIT*

*NO AP credit for  
calculator or syntax*

Problems 5 and 6 use the graph shown below.



A bug begins to crawl up a vertical wire at time  $t = 0$ . The velocity  $v$  of the bug at time  $t$ ,  $0 \leq t \leq 8$ , is given by the function whose graph is shown above.

5. (Non-Calculator) At what value of  $t$  does the bug change direction?

(A) 2

(B) 4

(C) 6

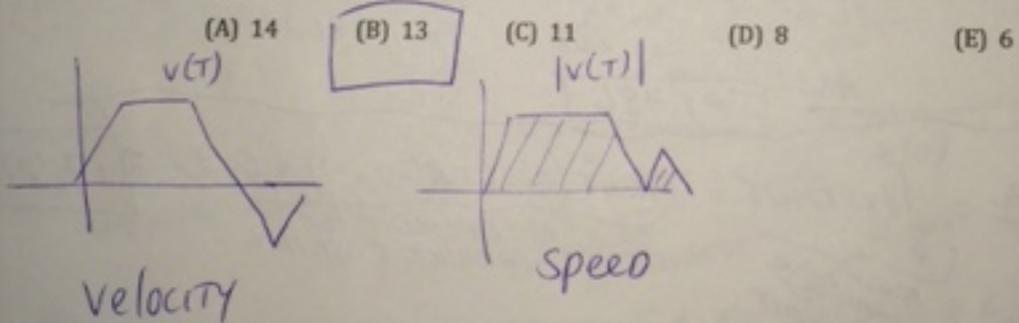
(D) 7

(E) 8

+this means (the given)  
 $v(t)$  changes sign...

meaning  
the given graph  
crosses the  
x-axis

6. (Non-Calculator) What is the total distance the bug traveled from  $t = 0$  to  $t = 8$ ?



$$\text{TOTAL DISTANCE} = \frac{2}{2} + \frac{1}{2}$$

$$= \frac{1}{2}(3)(2+6) + \frac{1}{2}(2)(1) = 12 + 1 = 13$$