APBC - U4 review ANS

1) $\frac{dy}{dx} = \sin x \cos^2 x \qquad y\left(\frac{\pi}{2}\right) = 0$

$\int dy = \int \sin x \cos^2 x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$\frac{du}{-\sin x} = dx$

$y = \int \cancel{\sin x} \cdot u^2 \cdot \frac{du}{-\cancel{\sin x}}$

$y = -\int u^2 du = -\frac{u^3}{3} + C = -\frac{\cos^3 x}{3} + C$

$0 = -\frac{(\cos \pi/2)^3}{3} + C$

$0 = 0 + C$

$C = 0$

$\boxed{y = -\frac{\cos^3 x}{3}}$

so $y(0) =$

$y(0) = -\frac{(\cos 0)^3}{3}$

$= \boxed{-1/3}$

B

~~this is A~~

APBC - 44 review solns

(2) $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$

$\frac{dP}{dt} = P\left(2\left(1 - \frac{P}{10000}\right)\right)$

$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10000}\right) \leftarrow$ this is a logistic curve

$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$

K is the carrying capacity...

$\boxed{\text{thus } \lim_{t\to\infty} P(t) = K = 10000}$ (E)

(3) $\frac{dy}{dx} = x^2 y$

$\int \frac{dy}{y} = \int x^2 dx$

$\ln|y| = \frac{x^3}{3} + C$

$y = e^{x^3/3 + C}$

$y = e^{x^3/3} \cdot \underbrace{e^C}_{C_1}$

$\boxed{y = C_1 e^{x^3/3}}$

so (C)

APBC-U4 review soln's

④ $\frac{dy}{dt} = -2y$ $\quad y(0) = 1$ $\quad$ FIND t when $y = \frac{1}{2}$

$\frac{dy}{dt} = ky \rightarrow y = Ce^{kt}$

$y = Ce^{-2t}$

$y(0) = 1 = Ce^{0}$

$C = 1$

$\boxed{y = e^{-2T}}$ ← EQ...

NOW set $y = 1/2$

$\frac{1}{2} = e^{-2T}$

$\ln(1/2) = \ln e^{-2t}$ $\quad$ OR $\quad \ln(1/2) = -2t$

$\cancel{\ln 1} - \ln 2 = -2t$ $\quad\quad \frac{\ln(1/2)}{-2} = t$

0

$-\ln 2 = -2t$

NOT A choice so simplify $\ln(1/2)$

(C) $\boxed{\frac{\ln 2}{2} = t}$

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⑤ P = # people infected $\quad t$ = time in DAYS

$\frac{dP}{dT} = kP \longrightarrow P = Ce^{kt}$

$(0, 1000)$ so $C = 1000$

$P = 1000e^{kT}$

$(7, 1200)$

$1200 = 1000e^{k(7)}$

$\frac{1200}{1000} = e^{7k}$

$\frac{6}{5} = e^{7k}$

$\ln\frac{6}{5} = 7k$

$\frac{1}{7}\ln(6/5) = k$

APBC - U4 review Sol/NS

5 (CONT)

$$P = 1000\, e^{\left(\frac{1}{7}\ln\frac{6}{5}\right)T}$$

at t=12

$$P = 1000\, e^{\frac{12}{7}\ln(6/5)}$$

$$P = 1000\, \cancel{e}^{\cancel{\ln}(6/5)^{12/7}}$$

$$P = 1000\,(6/5)^{12/7}$$

so I used 1993 rules (TYPEY - TYPEY - TYPEY)

* NOTE, this problem is from 1993... IN 1993 you could use A SCIENTIFIC (NON-graphing) CALCULATOR

$P \approx 1366.907$ (8)

$\approx \boxed{1367}$ C

You may see A problem of this type on the U4 exam and the AP EXAM... but IT will be solveable w/o A calculator

NOTE: IF I had to estimate...
I would have SAID..

$$1000\,(6/5)^{12/7} < 1000\,(6/5)^{12/6}$$
$$< 1000\,(6/5)^2$$
$$< 1000\left(\frac{36}{25}\right)$$
$$< \frac{1000}{25}(36) = 40(36) =$$

$$1000(6/5)^{12/7} < 1440$$

so I would have guessed either 1367 or 1400 WAS most likely.

APBC-U4 review

6) $x=8$ $\quad \frac{dy}{dt} = \frac{1}{k}\frac{dx}{dt}$ for $y=\sqrt[3]{x}$

$y = x^{1/3}$

$\frac{dy}{dt} = \frac{1}{3}x^{-2/3}\frac{dx}{dT}$

$\frac{1}{k}\frac{dx}{dT} = \frac{1}{3x^{2/3}}\frac{dx}{dt}$

$\frac{1}{k} = \frac{1}{3x^{2/3}}$

$3x^{2/3} = k$

when $x=8$...

so

$3(8)^{2/3} = k$

$3(8^{1/3})^2 = k$

$3(2^2) = k$

$3(4) = k$

$\boxed{12 = k}$

E

7) the slope field is symmetric about the y-axis... so the solution curve has an $x^{\text{even power}}$... thus the non-integrated DIFF EQ must have an $x^{\text{odd power}}$ (NOT B or D)

the slope field is **NOT** symmetric about the x-axis.. so soln curve Does not have y^{even} (so it has y^{odd})... thus the non-integrated DIFF. EQ has y^{even}... (NOT A, C, D)

the only answer that has

$$\frac{dy}{dx} = \frac{x^{\text{odd}}}{y^{\text{even}}} \text{ is } \text{E}$$

(NOTE: $\underbrace{\int y^{\text{even}}dy}_{\text{becomes odd}} = \underbrace{\int x^{\text{odd}}dx}_{\text{becomes even}}$)

APBC - U4 review solns

(8) $y = x + \frac{k}{x} = x + kx^{-1}$

NOTE: VA @ $x=0$

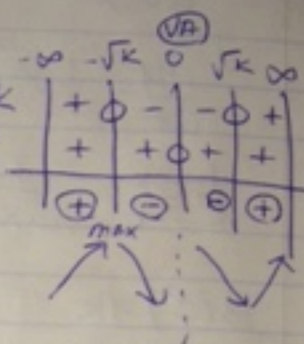
$y' = 1 - kx^{-2} = 1 - \frac{k}{x^2} = \frac{x^2-k}{x^2}$

CRIT pts:

$0 = x^2 - k \qquad 0 = x^2$

$x^2 = k \qquad x = 0$

$x = \pm\sqrt{k}$



max happens @
$x = -\sqrt{k}$

they say max happens @ $x = -2$

$-2 = -\sqrt{k}$

$2 = \sqrt{k}$

$2^2 = k$

$\boxed{k = 4}$ (D)

(9) $h(x) = f^2(x) - g^2(x) = [f(x)]^2 - [g(x)]^2$

requires chain rule   requires chain rule

$h'(x) = 2[f(x)]^1 \cdot f'(x) - 2[g(x)]^1 \cdot g'(x)$

$h'(x) = 2f(x)f'(x) - 2g(x)g'(x)$

given $f'(x) = -g(x)$ given $g'(x) = f(x)$

$h'(x) = 2f(x) \cdot -g(x) - 2g(x)f(x)$

$= -2f(x)g(x) - 2f(x)g(x)$

$= \boxed{-4f(x)g(x)}$ (C)

APBC -
U4 review solNS

(10) $\frac{dP}{dT} = \frac{P}{5}\left(1 - \frac{P}{12}\right)$ is a logistic curve

12 = carrying capacity

(A) It is irrelevant what $P(0)$ is equal to... For a logistic curve, the $\lim_{t\to\infty} P(t)$ = the carrying capacity

thus for both, the answer is

$$\boxed{\lim_{t\to\infty} P(t) = 12}$$

(B) For ANY logistic curve, the population grows fastest when the population is half the carrying capacity...
thus $P = \frac{12}{2} = \boxed{6}$

(C) $\frac{dy}{dt} = \frac{y}{5}\left(1 - \frac{t}{12}\right)$

$$\int 5\frac{dy}{y} = \int\left(1 - \frac{t}{12}\right)dt$$

$$5\ln|y| = t - \frac{t^2}{24} + C$$

$C/5$

$$\ln|y| = \frac{t}{5} - \frac{t^2}{120} + C_1$$

$$y = e^{t/5 - t^2/120} \cdot \underbrace{e^{C_1}}_{C_2}$$

$$y = C_2 e^{t/5 - t^2/120}$$

$y(0) = 3$

$3 = C_2 e^0$

$3 = C_2$

$$\boxed{y = 3e^{t/5 - \frac{t^2}{120}}}$$

(D) $\lim_{t\to\infty} 3e^{T/5 - \frac{T^2}{120}}$

$= \lim_{t\to\infty} 3e^{\frac{24T - T^2}{120}}$

$= \lim_{t\to\infty} 3e^{\frac{-T^2 + 24T}{120}}$

$= \boxed{0}$

$\lim_{t\to\infty} \frac{-T^2 + 24}{120} = \boxed{-\infty}$

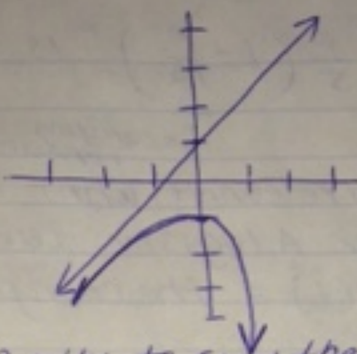
so $3e^{-\infty} = \frac{3}{e^{\infty}}$

APBC- U4 review solNs

(11)

$\frac{dy}{dx} = 2y - 4x$

(A) (Draw it on the slope field)



(B) we use $\frac{dy}{dx} = 2y - 4x$ to find the slope....

we start at the given point (0,1).
then we use point slope to find the new y-value for the next point... and so on...

$y - y_1 = m(\underbrace{x - x_1}_{\text{size of step}})$ so $y = m(\Delta x) + y_1$

$\Delta x = .1$ (given)

I do it in a table to keep my #s organized (work →)

x	y	$m = 2y - 4x$	WORK $y = m(\Delta x) + y_1$	New y
0	1	$m = 2(1) - 4(0) = 2$	$y = 2(.1) + 1 = 1.2$	1.2
.1	1.2	$m = 2(1.2) - 4(.1) = 2.4 - .4 = 2$	$y = 2(.1) + 1.2 = .2 + 1.2$	1.4
.2	1.4			

Δx <

↑ answer

$f(.2) \approx 1.4$

(C) $y = 2x + b$ so

$\frac{dy}{dx} = 2$

but we know

$\frac{dy}{dx} = 2y - 4x$

so $2 = 2y - 4x$

$4x + 2 = 2y$

$\boxed{2x + 1 = y}$

$y = 2x + b$ AND $y = 2x + 1$

$\boxed{b = 1}$

APBC Unit 4 Review Solutions—AP Scoring Guidelines

10.

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3$, $\lim_{t \to \infty} P(t) = 12$. If $P(0) = 20$, $\lim_{t \to \infty} P(t) = 12$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y \rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables
(d) $\lim_{t \to \infty} Y(t) = 0$	1 : answer $0/1$ if Y is not exponential

APBC-U4 review sol'ns

(1) (D) $g(0)=0$

$$\frac{dy}{dx} = 2y-4x$$

$$\left.\frac{dy}{dx}\right|_{(0,0)} = 2(0)-4(0)=0$$

so

$g'(0,0) = 0$ ← so $(0,0)$ is A critical point...

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we CAN use d^2y/dx^2 TO DeTermine IF IT'S A MAX or A MIN...

IF $\frac{d^2y}{dx^2} < 0$ at $(0,0)$ ← max

IF $\frac{d^2y}{dx^2} > 0$ at $(0,0)$ ← min

$$\frac{dy}{dx} = 2y-4x$$

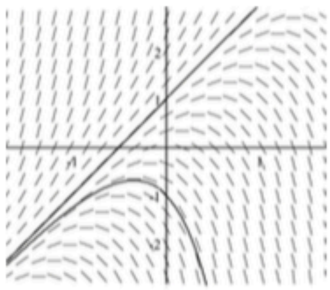
$$\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4$$ b/c we already know $\left.\frac{dy}{dx}\right|_{(0,0)} = 0$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,0)} = 2(0)-4 = -4$$ ← so $\left.\frac{d^2y}{dx^2}\right|_{(0,0)} < 0$

thus $g(0)=0$ is A $\boxed{\text{local MAXIMUM}}$

11.

(a)



$2\begin{cases} 1: \text{ solution curve through } (0,1) \\ 1: \text{ solution curve through } (0,-1) \end{cases}$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 1 + (2-0)(0.1) = 1.2$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

$2\begin{cases} 1: \text{ Euler's method equations or equivalent table applied to (at least) two iterations} \\ 1: \text{ Euler approximation to } f(0.2) \text{ (not eligible without first point)} \end{cases}$

(c) Substitute $y = 2x + b$ in the DE:
$2 = 2(2x+b) - 4x = 2b$, so $b = 1$

OR

Guess $b = 1$, $y = 2x + 1$

Verify: $2y - 4x = (4x+2) - 4x = 2 = \dfrac{dy}{dx}$.

$2\begin{cases} 1: \text{ uses } \dfrac{d}{dx}(2x+b) = 2 \text{ in DE} \\ 1: \ b = 1 \end{cases}$

(d) g has local maximum at $(0,0)$.

$g'(0) = \left.\dfrac{dy}{dx}\right|_{(0,0)} = 2(0) - 4(0) = 0$, and

$g''(x) = \dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - 4$, so

$g''(0) = 2\,g'(0) - 4 = -4 < 0$.

$3\begin{cases} 1: \ g'(0) = 0 \\ 1: \text{ shows } g''(0) = -4 \\ 1: \text{ conclusion} \end{cases}$