APBC - U6 Review Sol'ns (p.1)

① which is convergent?

I. $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}+\ldots$ p-series w/ $p=2$ CONVERGES

II. $1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}+\ldots$ p-series w/ $p=1$ DIVERGES (it's the harmonic series)

III. $1-\frac{1}{3}+\frac{1}{3^2}+\ldots\frac{(-1)^{n+1}}{3^{n-1}}+\ldots$ this is a geometric series with $a=1$, $r=-\frac{1}{3}$

$\underset{a}{1}\,\underset{a}{1}\underset{r}{\left(-\frac{1}{3}\right)}^1+\underset{a}{1}\underset{r^2}{\left(-\frac{1}{3}\right)^2}+\ldots$

since $|r|=\left|-\frac{1}{3}\right|<1$

this converges

$\boxed{\text{I and III}}$ ⓒ (to $\frac{a}{1-r}=\frac{1}{1+\frac{1}{3}}=\frac{3}{4}$ ... but they DIDN'T ASK)

② $\sin t \approx t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}\ldots$ ← memorized MacLaurin series

so $\frac{\sin t}{t}\approx\frac{t}{t}-\frac{\frac{t^3}{3!}}{t}+\frac{\frac{t^5}{5!}}{t}-\frac{\frac{t^7}{7!}}{t}+\ldots$

$\approx 1-\frac{t^2}{3!}+\frac{t^4}{5!}-\frac{t^6}{7!}+\ldots$ Ⓐ

③ $\sum_{k=1}^{\infty}\frac{(x+1)^k}{k^2}$ use the Ratio test to find the interval of convergence then test both endpoints to see if it converges or diverges at each endpoint

$$\lim_{k\to\infty}\left|\frac{\frac{(x+1)^{k+1}}{(k+1)^2}}{\frac{(x+1)^k}{k^2}}\right|<1$$

$$\lim_{k\to\infty}\left|\frac{(x+1)^{k+1}}{(k+1)^2}\cdot\frac{k^2}{(x+1)^k}\right|<1$$

$$\lim_{k\to\infty}\left|\frac{(x+1)^{k+1}}{(x+1)^k}\cdot\frac{k^2}{(k+1)^2}\right|<1$$

$$=\lim_{k\to\infty}\left|(x+1)\,\frac{k^2}{k^2+2k+1}\right|<1$$

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P.2

(3) (cont.)

$$\lim_{k\to\infty}\left|(x+1)\left(\frac{k^2}{k^2+2k+1}\right)\right|<1$$

$$\underbrace{\lim_{k\to\infty}\left|\frac{k^2}{k^2+2k+1}\right|}\cdot|x+1|<1$$

$$1\cdot|x+1|<1$$

$$|x+1|<1$$

$x+1<1$ and $x+1>-1$

$x<0$ and $x>-2$] so C, D, E (A, B have wrong #s)

Now test end points

@ $x=0$

$$\sum_{k=1}^{\infty}\frac{(0+1)^k}{k^2}=\sum_{k=1}^{\infty}\frac{1}{k^2}=\text{convergent (p-series w/ } p=2)$$

@ $x=-2$

$$\sum_{k=1}^{\infty}\frac{(-2+1)^k}{k^2}=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}=\text{convergent (p-series w/ } p=2 \text{ and alternating)}$$

So ANS: $[-2,0]$ aka $\boxed{-2\le x\le 0}$ (E)

(4) $f(x)=\ln(3-x)$ $\qquad f(2)=\ln(3-2)=0$

$f'(x)=\frac{-1}{3-x}=-1(3-x)^{-1}$ $\qquad f'(2)=\frac{-1}{3-2}=-1$

$f''(x)=1(3-x)^{-2}(-1)=\frac{-1}{(3-x)^2}$ $\qquad f''(2)=\frac{-1}{(3-2)^2}=-1$

$f'''(x)=2(3-x)^{-3}(-1)=\frac{-2}{(3-x)^3}$ $\qquad f'''(2)=\frac{-2}{(3-2)^3}=-2$

$$P_3(x)=f(2)+f'(2)(x-2)+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3$$

$$=\varnothing-1(x-2)+\frac{-1}{2}(x-2)^2+\frac{-2}{6}(x-2)^3$$

$$=\boxed{-1(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}}$$ (B)

⑤ $f(x) = e^{3x}$

recall $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(E)

so $e^{3x} \approx 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots$

so $\frac{(3x)^3}{3!} = \frac{27x^3}{6} = \frac{9x^3}{2}$ $\boxed{\text{coefficient of } x^3 \text{ is } 9/2}$

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⑥ $\sum_{n=0}^{\infty} a_n x^n = f(x)$

thus $f'(x) = \sum_{n=0}^{\infty} n \cdot a_n x^{n-1}$

$f'(1) = \sum_{n=0}^{\infty} n \cdot a_n (1)^{n-1} = \sum_{n=0}^{\infty} n a_n$ ← but this is NOT A choice... so you need to think about why...

well $\sum_{n=0}^{\infty} n a_n = \underbrace{\cancel{0} a_0}_{\text{when } n=0 \text{ the term}=0} + \sum_{n=1}^{\infty} n a_n = \boxed{\sum_{n=1}^{\infty} n a_n}$ (D)

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⑦ which Diverge?

(I) $\sum_{k=3}^{\infty} \frac{2}{k^2+1} < \sum_{k=3}^{\infty} \frac{2}{k^2}$ ← which converges because $2\sum_{k=2}^{\infty} \frac{1}{k^2}$ is p-series with $p = 2$

by Direct comparison, this series Converges

(II) $\sum_{k=1}^{\infty} \left(\frac{6}{7}\right)^k = \underset{a}{\frac{6}{7}} + \underset{ar}{\frac{6}{7}\left(\frac{6}{7}\right)} + \underset{ar^2}{\frac{6}{7}\left(\frac{6}{7}\right)^2} + \dots$ geometric series with $r = 6/7$

$\left|\frac{6}{7}\right| < 1$ so IT converges

SO IT's either "none" OR "III only"

(continued on next page)

(7) (CON.T.)

(III) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$

IT's the alternating harmonic series which converges

* you should memorize that this converges... but you can also show IT using the alternating series test.

$\sum_{k=2}^{\infty} \underbrace{(-1)^k}_{\text{Alt}} \underbrace{\left(\frac{1}{k}\right)}_{b_n}$

✓ $b_{n+1} < b_n$ (b is Decreasing)

✓ $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

SO IT converges

thus the answer is (A) NONE of them Diverge

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(8) $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$ use ratio test to find interval of convergence then test the endpts to see if they converge

$$\lim_{n\to\infty} \left| \frac{\frac{(x-1)^{n+1}}{3^{n+1}}}{\frac{(x-1)^n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \cdot \frac{(x-1)^{n+1}}{(x-1)^n} \right| = \lim_{n\to\infty} \left| \frac{1}{3} \cdot (x-1) \right|$$

there's no "n" left... so we DON'T Need the limit anymore

$$= \lim_{n\to\infty} \left| \tfrac{1}{3}(x-1) \right| = \left| \tfrac{1}{3}(x-1) \right| < 1$$

$\frac{1}{3}(x-1) < 1$ and $\frac{1}{3}(x-1) > -1$

$x-1 < 3$ $\qquad$ $x-1 > -3$

$x < 4$ and $x > -2$

SO IT's C or D and we only need to test $x=-2$

@ $x=-2$

$$\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{3^n}}{\cancel{3^n}} = \sum_{n=0}^{\infty} (-1)^n$$

Continued on next page

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8 (CONT.)

@ $x=-2$ (we DON'T have to check $x=4$ b/c both C and D DO NOT include $x=4$)

$$\sum_{n=0}^{\infty}(-1)^n = \underbrace{1+-1+1+-1+}\ldots$$

this series DIVERGES because the partial sums oscillate between 0 and 1

$$\left.\begin{aligned} S_0 &= 1 \\ S_1 &= 1+-1=0 \\ S_2 &= 1+-1+1=1 \end{aligned}\right\} \text{forever}$$

C

SO, ANS $(-2,4)$ aKA $\boxed{-2<x<4}$

9 $S_n = \left(\frac{(5+n)^{100}}{5^{n+1}}\right)\left(\frac{5^n}{(4+n)^{100}}\right)$

$$= \frac{(5+n)^{100}}{(4+n)^{100}} \cdot \frac{5^n}{5^{n+1}} = \frac{(5+n)^{100}}{(4+n)^{100}} \cdot \frac{1}{5}$$

$$= \frac{(n+5)^{100}}{(n+4)^{100}} \cdot \frac{1}{5}$$ ← when we talk about convergence, we're talking about as $n \to \infty$

$$S_\infty = \lim_{n\to\infty} \frac{(n+5)^{100}}{(n+4)^{100}} \cdot \frac{1}{5} = \frac{1}{5} \lim_{n\to\infty} \frac{(n+5)^{100}}{(n+4)^{100}}$$

For INFINITE VALUES OF n, we only care about the LEADING terms..

$(n+5)^{100} = n^{100} +$ other n-stuff

$(n+4)^{100} = n^{100} +$ other n-stuff

$S_\infty = \frac{1}{5}(1) = \frac{1}{5}$ A

APBC U6 review solns (p.6) FR 10

(A) $P_1(x) = f(0) + f'(0)(x)$

$P_1(\tfrac{1}{2}) = f(0) + f'(0)(\tfrac{1}{2}) = -3$

$-4 + f'(0)\cdot\tfrac{1}{2} = -3$

$f'(0)\cdot\tfrac{1}{2} = -3+4$

$\tfrac{1}{2}f'(0) = 1$ TADA ☺

$\boxed{f'(0)=2}$

(B) $P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$

$P_3(x) = -4 + 2x + \frac{(-2/3)}{2}x^2 + \frac{(1/3)}{3!}x^3$

$\boxed{P_3(x) = -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3}$

(C) $h'(x) = f(2x)$

$h''(x) = f'(2x)\cdot 2 = 2f'(2x)$

$h'''(x) = 2f''(2x)\cdot 2 = 4f''(2x)$

to write 3rd degree Taylor Polynomial for $H(x)$, we need up to $h'''(x)$

we used $P_3(x)$ for $f(x)$ poly...
So pick a Different letter for $h(x)$ poly.
I pick Q

$Q_3(x) = h(0) + h'(0)x + \frac{h''(0)}{2!}x^2 + \frac{h'''(0)}{3!}x^3$

$Q_3(x) = 7 + f(2\cdot 0)x + \frac{2f'(2\cdot 0)}{2!}x^2 + \frac{4f''(2\cdot 0)}{3!}x^3$

$Q_3(x) = 7 + f(0)\cdot x + f'(0)x^2 + \frac{4}{6}f''(0)x^3$

$Q_3(x) = 7 + -4x + 2x^2 + \frac{2}{3}(-\frac{2}{3})x^3$

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FRQ (C) (CONT.)

$$\boxed{Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3}$$

11FR let $f(x) = \ln(1+x^3)$

(A) $f(x) = (x^3) - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \ldots + (-1)^{n+1}\frac{x^{3n}}{n}$

$$\boxed{= x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots (-1)^{n+1}\frac{x^{3n}}{n}}$$

(B) use the ratio test...

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}x^{3(n+1)}}{n+1}}{\frac{(-1)^{n+1}x^{3n}}{n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(-1)^{n+1}}\cdot\frac{x^{3n+3}}{x^{3n}}\cdot\frac{n}{n+1}\right|$$

this comes out to -1... which doesn't matter b/c of the absolute value

$$= \lim_{n\to\infty}\left|x^3\left(\frac{n}{n+1}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{n}{n+1}\right|\cdot|x^3| < 1$$

$$1\cdot|x^3| < 1$$

$$|x^3| < 1$$

$x^3 < 1$ and $x^3 > -1$

$x < 1$ and $x > -1$

Now test endpoints

@ $x = 1$: $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}$ ← alternating harmonic series converges

@ $x = -1$: $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(-1)^{3n}}{n} = \sum_{n=0}^{\infty}\frac{(-1)^{4n+1}}{n}$ ← $(-1)^{odd} = -1$ ← top is always ⊖ so NON-alternating harmonic

Diverges

ANS: $(-1, 1]$ or $-1 < x \leq 1$

FR11 (continued)

C $f(x)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{3n}}{n}$

$F'(x)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{3n}{n}x^{3n-1}=\sum_{n=0}^{\infty}(-1)^{n+1}3\cdot x^{3n-1}$

$f'(T^2)=\sum_{n=1}^{\infty}(-1)^{n+1}\cdot 3\,(T^2)^{3n-1}=\sum_{n=1}^{\infty}(-1)^{n+1}\cdot 3(T^{6n-2})$

$F'(T^2)=(-1)^2\cdot 3T^4+(-1)^3\cdot 3T^{10}+(-1)^4(3)T^{16}+(-1)^5(3)T^{22}$

$\boxed{F'(T^2)=3T^4-3T^{10}+3T^{16}-3T^{22}}$

$g(x)=\int_0^x F'(T^2)\,dT$

$g(x)\approx\frac{3}{5}T^5-\frac{3}{11}T^{11}$

$g(1)\approx\frac{3}{5}(1^5)-\frac{3}{11}(1)^{11}=\frac{3}{5}-\frac{3}{11}=\frac{33}{55}-\frac{15}{55}\approx\boxed{\frac{18}{55}}$

D since we used the 1st two terms to find the value in Part C, we use the Next term to find the error...

$F'(T^2)=\sum_{n=1}^{\infty}(-1)^{n+1}3(T^{6n-2})$

$G(x)=\int_0^x F'(T^2)\,dT=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\cdot T^{6n-1}}{6n-1}$

the 3rd term in the series is

$\frac{g^{(3)}(a)(x)^3}{3!}$ (see Next page...)

APBC - U6 review soln

$$g(x) = \underbrace{\frac{3}{5}T^5 - \frac{3}{11}T^{11}}_{\text{Approximation used these two terms}} + \frac{3}{17}T^{17} + \frac{-3}{23}T^{23} \dots$$

So error will use the next term

$$E = \left| g(1) - \frac{18}{55} \right|$$

Actual estimate

$$E < \frac{3(1)^{17}}{17} = \frac{3}{17} < \frac{3}{15} = \frac{1}{5}$$

thus

$$E < \frac{3}{17} < \frac{1}{5}$$

$$\left| g(1) - 18/55 \right| < 1/5$$

Question 6

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$$P_1\left(\frac{1}{2}\right) = -4 + f'(0) \cdot \frac{1}{2} = -3$$
$$f'(0) \cdot \frac{1}{2} = 1$$
$$f'(0) = 2$$

$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$

(b) $$P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right) \cdot \frac{x^2}{2!} + \frac{1}{3} \cdot \frac{x^3}{3!}$$
$$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$$
$$Q_3(x) = -4x + 4 \cdot \frac{x^2}{2} - \frac{4}{3} \cdot \frac{x^3}{3} + C;\ C = Q_3(0) = h(0) = 7$$
$$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

OR

$$h'(x) = f(2x),\ h''(x) = 2f'(2x),\ h'''(x) = 4f''(2x)$$
$$h'(0) = f(0) = -4,\ h''(0) = 2f'(0) = 4,\ h'''(0) = 4f''(0) = -\frac{8}{3}$$
$$Q_3(x) = 7 - 4x + 4 \cdot \frac{x^2}{2!} - \frac{8}{3} \cdot \frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$

Question 6

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) The interval of convergence is centered at $x = 0$.

At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges.

At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges.

Therefore the interval of convergence is $-1 < x \le 1$.

2 : answer with analysis

(c) The Maclaurin series for $f'(x)$, $f'\left(t^2\right)$, and $g(x)$ are

$$f'(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$$

$$f'\left(t^2\right) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$$

$$g(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$$

Thus $g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$.

$4 : \begin{cases} 1 : \text{two terms for } f'\left(t^2\right) \\ 1 : \text{other terms for } f'\left(t^2\right) \\ 1 : \text{first two terms for } g(x) \\ 1 : \text{approximation} \end{cases}$

(d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0.

Thus $\left| g(1) - \frac{18}{55} \right| < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}$.

1 : analysis