

$$(1) \sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$$

DIVERGES

Divergence test

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2-1}{n^2+n} = 1 \neq 0$$

$$(2) \sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$$

Divergence test is inconclusive $\lim_{n \rightarrow \infty} a_n = 0$.

not p-series. not geometric. not alternating.

try partial fractions...

$$\frac{n-1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1} = \frac{-1}{n} + \frac{2}{n+1} \leftarrow \text{not telescoping in the traditional sense... but maybe we can use it in a less conventional way}$$

$$n-1 = a(n+1) + bn$$

$$n=0: -1 = a$$

$$n=-1: -2 = b(-1) \\ a=b$$

$$\sum_{n=1}^{\infty} \frac{-1}{n} + \frac{2}{n+1} = \left(\frac{-1}{1} + \frac{2}{2} \right) + \left(\frac{-1}{2} + \frac{2}{3} \right) + \left(\frac{-1}{3} + \frac{2}{4} \right) + \left(\frac{-1}{4} + \frac{2}{5} \right) \dots$$

$$= -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \dots$$

$$= -1 + \sum_{n=2}^{\infty} \frac{1}{n} \leftarrow \text{this is the harmonic series and it's divergent}$$

$$= -1 + \text{Divergent series} = \text{Divergent}$$

* this is a different way from the online soln but should work fine.

$$(3) \sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

Divergence test is inconclusive

$$\lim_{n \rightarrow \infty} \frac{1}{n^2+n} = 0$$

not p-series, not geometric not alternating,

try partial fractions to check for telescoping...

$$\frac{1}{n(n+1)} = \frac{a}{n} + \frac{b}{n+1} = \frac{+1}{n} + \frac{-1}{n+1} \quad \leftarrow \leftarrow \text{Yes!! IT telescopes ...}$$

$$1 = a(n+1) + b(n)$$

$$\underline{n=0}: 1 = a \quad \underline{n=-1}: 1 = -b$$
$$-1 = b$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n} + \frac{-1}{n+1} = \left(\frac{1}{1} + \frac{-1}{2} \right) + \left(\frac{1}{2} + \frac{-1}{3} \right) + \left(\frac{1}{3} + \frac{-1}{4} \right) \dots$$

sum to 0 sum to 0 etc...

$$= \boxed{1} \leftarrow \text{b/c all other terms sum (in pairs) to 0 ...}$$

Converges

$$(4) \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{n-1}{n^2+n} \right)$$

Divergence test is inconclusive

$$\lim_{n \rightarrow \infty} (-1)^{n-1} \left(\frac{n-1}{n^2+n} \right) = (-1)^{\infty-1} (0) = 0$$

$(-1)^{n-1}$ is a hint to try alternating series...

$$(1) a_n = (-1)^n b_n \text{ or } a_n = (-1)^{n \pm 1} (b_n)$$

(2) b_n is decreasing and positive and trends to 0 ...

$$b_n \geq 0$$
$$b_{n+1} \leq b_n$$
$$\lim_{n \rightarrow \infty} b_n = 0$$

IF SO (both conditions) then the series converges...

$$(4) \text{ (cont.) } \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{n-1}{n^2+n} \right)$$

↑
~~b_n~~ b_n

$$(1) a_n = (-1)^{n-1} b_n \checkmark$$

$$(2) b_n \geq 0 \leftarrow \text{true ... for } n \geq 1 \quad \begin{matrix} n-1 = (1), (2) \\ n^2+n = (2) \end{matrix} \checkmark$$

(3) b_n is decreasing...

$$\begin{matrix} n=1 & n=2 & n=3 & n=4 \\ \frac{0}{1+1} = 0 & \frac{1}{4+2} = \frac{1}{6} & \frac{2}{9+3} = \frac{2}{12} = \frac{1}{6} & \frac{3}{16+4} = \frac{3}{20} \end{matrix} \quad \text{yes...} \checkmark$$

$$(4) \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n-1}{n^2+n} = 0 \quad \text{yes...} \checkmark$$

so converges

$$(5) \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{2n}} = \sum_{n=1}^{\infty} \frac{(-1 \cdot 3)^{n+1}}{8^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n+1}}{8^n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{9 \cdot 3^{n-1}}{8 \cdot 8^{n-1}} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{9}{8} \right) \left(\frac{3}{8} \right)^{n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^2 (-1)^{n-1} \left(\frac{9}{8} \right) \left(\frac{3}{8} \right)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{9}{8} \right) \left(\frac{3}{8} \right)^{n-1}$$

↑
alternating series

$$\begin{matrix} b_n \geq 0 \\ b_{n+1} \leq b_n \\ \lim_{n \rightarrow \infty} b_n = 0 \end{matrix}$$

so converges by the alternating series test.

side note: b_n is actually a geometric series... so if not for the (-1)ⁿ⁻¹, this series would still converge.

* this problem can also be done w/ the ratio test... (see next page)...

$$\textcircled{5} \sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}} \quad (\text{using the ratio test})$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \frac{2^{3n}}{(-3)^{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+2-(n+1)}}{2^{3n+3-3n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^1}{2^3} \right| = \frac{3}{8} < 1 \end{aligned}$$

so it **converges** (absolutely)

$$\textcircled{6} \sum_{n=1}^{\infty} \left(\frac{3n}{1+8n} \right)^n \quad \text{Divergence test}$$

$$\lim_{n \rightarrow \infty} \left(\frac{3n}{1+8n} \right)^n = \left(\frac{3}{8} \right)^{\infty} = 0 \quad \text{inconclusive}$$

keep going... try comparison test:

$$\frac{3n}{1+8n} \leq \frac{3n}{8n} = \frac{3}{8}$$

$$\text{so } a_n = \left(\frac{3n}{1+8n} \right)^n \leq \left(\frac{3}{8} \right)^n = b_n$$

$\sum_{n=1}^{\infty} \left(\frac{3}{8} \right)^n$ is a geometric sequence that converges (since $|r| < 1$)

by the comparison test, it **converges**

~~try comparison test~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^2}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^3}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^4}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^5}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^6}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^7}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^8}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^9}$~~
 ~~$\sum_{n=1}^{\infty} \frac{1}{n^{10}}$~~

$$\textcircled{7} \sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

Divergence test is inconclusive

$$\lim_{n \rightarrow \infty} \frac{1}{n\sqrt{\ln n}} = 0$$

not p-series, not alternating, not geo.
not telescoping (breaks into ugly partial fractions.) not Taylor.

Comparison test and limit comparison test were tough (tried $\frac{1}{n}$ for limit comparison but it was inconclusive.)

try integral test (b/c if $u = \ln x$, $du = \frac{1}{x} dx$ which is in the integral)

$$a_n = f(n) \dots f(x) = \frac{1}{x\sqrt{\ln x}}$$

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}} dx = \int \frac{1}{x\sqrt{u}} \cdot x du = \int u^{-1/2} du = [2u^{1/2}]$$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$x du = dx$$

$$= \lim_{n \rightarrow \infty} [2\sqrt{\ln x}]_2^n = \lim_{n \rightarrow \infty} 2\sqrt{\ln n} - 2\sqrt{\ln 2} = \infty \text{ Diverges}$$

so, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ Diverges

$$\textcircled{8} \sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!} \quad \text{try divergence test...}$$

$$\lim_{k \rightarrow \infty} \frac{2^k k!}{(k+2)!} = \lim_{k \rightarrow \infty} \frac{2^k \cdot k!}{k! (k+1)(k+2)} = \lim_{k \rightarrow \infty} \frac{2^k}{k^2 + 3k + 2}$$

$= \frac{\infty}{\infty}$ which is indeterminate... so you can use L'Hopital's...

$$\lim_{k \rightarrow \infty} \frac{2^k}{k^2 + 3k + 2} = \lim_{k \rightarrow \infty} \frac{\ln 2 \cdot 2^k}{2k + 3} = \lim_{k \rightarrow \infty} \frac{\ln 2 \cdot \ln 2 \cdot 2^k}{2} = \infty$$

still ∞/∞

so since $\lim_{k \rightarrow \infty} a_k = \infty$ it's DIVERGENT

$$(9) \sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$$

try the divergence test...

$$\lim_{k \rightarrow \infty} \frac{k^2}{e^k} = \frac{\infty}{\infty} = \lim_{k \rightarrow \infty} \frac{2k}{e^k} = \lim_{k \rightarrow \infty} \frac{2}{e^k} = 0$$

so use l'Hopital's still ∞

so inconclusive... keep using other tests...

try ratio test...

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\frac{(k+1)^2}{e^{k+1}}}{\frac{k^2}{e^k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{e^k (k^2 + 2k + 1)}{e^{k+1} \cdot k^2} \right| = \lim_{k \rightarrow \infty} \left| \frac{k^2 + 2k + 1}{k^2} \cdot \frac{1}{e} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \left(\frac{k^2}{k^2} + \frac{2k}{k^2} + \frac{1}{k^2} \right) \left(\frac{1}{e} \right) \right| = \lim_{k \rightarrow \infty} \left| \left(1 + \frac{2}{k} + \frac{1}{k^2} \right) \left(\frac{1}{e} \right) \right|$$

$$= \left| 1 \left(\frac{1}{e} \right) \right| = \frac{1}{e} < 1 \quad \text{so } \boxed{\text{Converges}} \\ \text{(absolutely)}$$

$$(10) \sum_{n=1}^{\infty} n^2 e^{-n^3} = \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$$

try the divergence test first...

$$\lim_{n \rightarrow \infty} \frac{n^2}{e^{n^3}} = \frac{\infty}{\infty} = \lim_{n \rightarrow \infty} \frac{2n}{3n^2 e^{n^3}} = \lim_{n \rightarrow \infty} \frac{2}{3n e^{n^3}} = \frac{1}{\infty} = 0$$

so use l'Hopital's

so inconclusive.

Next, I'd jump to the integral test b/c I see that if $u = -n^3$, $du = -3n^2 dn$ which is conveniently already part of an (well, n^2 is.)

$$a_n = n^2 e^{-n^3} = f(n)$$

$$\int_1^{\infty} x^2 e^{-x^3} dx = \int x^2 e^u \cdot \frac{du}{-3x^2} = -\frac{1}{3} \int e^u du = \left[-\frac{1}{3} e^u \right] \\ u = -x^3 \quad du = -3x^2 dx = \left[-\frac{1}{3} e^{-x^3} \right]_1^{\infty} = \lim_{n \rightarrow \infty} \left[-\frac{1}{3} e^{-x^3} \right]_1^n$$

⑩ (continued)

$$= \lim_{n \rightarrow \infty} \left[-\frac{1}{3} e^{-x^3} \right]_1^n$$

$$= \lim_{n \rightarrow \infty} \left(-\frac{1}{3} e^{-n^3} + \frac{1}{3} e^{-1} \right) = \lim_{n \rightarrow \infty} \left(\frac{-1}{3e^{n^3}} + \frac{1}{3e} \right)$$

$= 0 + \frac{1}{3e} = \frac{1}{3e}$... so it converges
which means

$$\sum_{n=1}^{\infty} n^2 e^{-n^3} \quad \boxed{\text{converges}}$$

11) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

Divergence test

$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}}{n \ln n} = \frac{\pm 1}{\infty} = 0$$

so inconclusive...

the $(-1)^{n+1}$ is a hint to look at alternating series test first...

NOTE that: $(-1)^{n+1} = (-1)^2 \cdot (-1)^{n-1} = (-1)^{n-1}$ all the time (so even if you see $(-1)^{n+1}$... it works the same as $(-1)^{n-1}$)

$$\frac{(-1)^{n+1}}{n \ln n} = (-1)^{n-1} \underbrace{\left(\frac{1}{n \ln n} \right)}_{b_n} = a_n$$

$$b_n \geq 0 \quad \checkmark$$

$$b_{n+1} \leq b_n \quad \checkmark$$

$$\lim_{n \rightarrow \infty} b_n = 0 \quad \checkmark$$

so by alternating series test it $\boxed{\text{converges}}$

$$(12) \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n^2+25} \right)$$

Divergence test is inconclusive
 $\lim_{n \rightarrow \infty} (-1)^n \left(\frac{n}{n^2+25} \right) = \neq 0$
 ← b/c denom power is bigger

try alternating series test since there's a $(-1)^n$

$$a_n = (-1)^n \underbrace{\left(\frac{n}{n^2+25} \right)}_{b_n}$$

$$b_n \geq 0 \checkmark$$

$$b_{n+1} \leq b_n \checkmark$$

$$\lim_{n \rightarrow \infty} b_n = 0 \checkmark$$

so it **converges** by alternating series test.

$$(13) \sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$$

the factorial is often a hint to use the ratio test...

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{3^{n+1} (n+1)^2}{(n+1)!}}{\frac{3^n n^2}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} (n+1)^2 \cdot n!}{(n+1)! \cdot 3^n n^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{3^{n+1} \cdot n! \cdot (n^2 + 2n + 1)}{3^n \cdot (n+1)! \cdot n^2} \right| = \lim_{n \rightarrow \infty} \left| \frac{3^1}{n+1} \cdot \left(\frac{n^2 + 2n + 1}{n^2} \right) \right|$$

$$= \lim_{n \rightarrow \infty} \left| \left(\frac{3}{n+1} \right) \cdot \left(\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2} \right) \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{3}{n+1} \right) \cdot \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) \right|$$

$$= \left(\frac{3}{\infty+1} \right) (1 + \phi + \phi) = \frac{3}{\infty} = 0 < 1$$

converges

(15) $\sum_{n=0}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}$ the factorial is a good hint to try the ratio test.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots (3n+2)(3(n+1)+2)}}{\frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n+2)}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots (3n+2)(3(n+1)+2)} \cdot \frac{2 \cdot 5 \cdot 8 \cdots (3n+2)}{n!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\cancel{n!} (n+1) \cdot 2 \cdot 5 \cdot 8 \cdots (3n+2)}{2 \cdot 5 \cdot 8 \cdots (3n+2)(3(n+1)+2) \cdot \cancel{n!}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n+1}{(3(n+1)+2)} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{3n+3+2} \right| =$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n+1}{3n+5} \right| = \frac{1}{3} \text{ use l'Hopital's}$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{3} \right| = \frac{1}{3} < 1 \quad \boxed{\text{Converges}}$$

(17) $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ use the divergence test...

$$\lim_{n \rightarrow \infty} (-1)^n 2^{1/n} = (\pm 1) 2^{1/\infty} = \pm 1(1) = \pm 1 \neq 0 \text{ so } \boxed{\text{DIVERGES}}$$

(19) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$ TRY Alternating series test...

$$b_n = \frac{\ln(n)}{\sqrt{n}}$$

① $b_n \geq 0$ ✓

② b_n is decreasing? Find derivative of b_n to determine if it's inc or dec...

(19) (cont.)

$$b_n = \frac{\ln n}{n^{1/2}}$$

$$b_n' = \frac{n^{1/2} \left(\frac{1}{n}\right) - \ln n \cdot \frac{1}{2} n^{-1/2}}{n}$$

$$= \frac{\frac{1}{n^{1/2}} - \frac{\ln n}{2n^{1/2}}}{n} = \frac{2 - \ln n}{2n^{3/2}}$$

$$b_n' = \frac{2 - \ln n}{n^{3/2}} \leftarrow \text{since } n \geq 1 \quad n^{3/2} = \oplus$$

$$\begin{array}{l} 2 - \ln n = 0 \\ 2 = \ln n \\ e^2 = n \end{array} \quad \begin{array}{c} e^2 \\ \oplus \mid \ominus \end{array} \quad \begin{array}{l} \text{so } b_n' \text{ is } \ominus \\ \text{After } n = e^2 \end{array}$$

meaning for all large values of n ,
 $b_{n+1} \leq b_n$ ✓

(3) is b_n treading to 0?

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n^{1/2}} = \frac{\infty}{\infty} \text{ use l'Hopital's}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{2} n^{-1/2}} = \lim_{n \rightarrow \infty} \frac{1/n}{1/(2n^{1/2})} = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{2n^{1/2}}{1}$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n^{1/2}} = 0 \quad \checkmark \text{ yes}$$

So, since b_n meets all three requirements for the alternating series test

$$\sum (-1)^n \frac{\ln n}{\sqrt{n}} \quad \boxed{\text{converges}}$$

$$(21) \sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \frac{((-2)^2)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n$$

use the n^{th} root test...

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{4}{n}\right)^n} = \lim_{n \rightarrow \infty} \frac{4}{n} = 0$$

since $0 < 1$, it **converges**

$$(23) \sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$$

this one is tough...
only way to do it
is limit comparison
test... use

$b_n = \frac{1}{n}$
(which is divergent,
FYI...)

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{\tan 0}{\frac{0}{\infty}} = \frac{0}{0} \leftarrow \text{indeterminate}$$

use l'Hopital's

$$\lim_{n \rightarrow \infty} \frac{\sec^2\left(\frac{1}{n}\right) \cdot \left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \sec^2\left(\frac{1}{n}\right)$$

$= \sec^2(0) = 1 = c > 0$ so by limit
comparison test, a_n and b_n

do the same thing...

b_n Diverges (it's the harmonic
series)

so $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ **Diverges**

too :)

$$(25) \sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$$

← Factorials and
n as exponent
SREAMS
RATIO test

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\frac{(k+1)!}{e^{(k+1)^2}}}{\frac{k!}{e^{k^2}}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{(k+1)!}{e^{k^2+2k+1}} \cdot \frac{e^{k^2}}{k!} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{(k+1) \cdot \cancel{k \cdots 1} \cdot e^{k^2}}{e^{k^2+2k+1} \cdot \cancel{k \cdots 1}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{k+1}{e^{k^2+2k+1-k^2}} \right| = \lim_{k \rightarrow \infty} \left| \frac{k+1}{e^{2k+1}} \right|$$

$$= \frac{\infty}{\infty} \text{ use l'Hopital's } \\ \text{(indeterminant form)}$$

$$= \lim_{k \rightarrow \infty} \left| \frac{1}{e^{2k+1} \cdot 2} \right| = \frac{1}{\infty} = 0 < 1 \quad \checkmark$$

Converges

$$(27) \sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$$

← TRICKY... requires a comparison, AND then the bn requires an integral test
YIKES!!

$$\approx \frac{k \ln k}{(k+1)^3} \leq \frac{k \ln k}{k^3} = \frac{\ln k}{k^2}$$

(b/c $(k+1) > k$ so bigger denom means smaller fraction)

So... if I can show this converges, then a_n must also converge (since it's smaller)

$$\sum_{k=1}^{\infty} \frac{\ln k}{k^2} = \sum_{k=1}^{\infty} \ln k \cdot \frac{1}{k^2}$$

use integral test (by parts!! wow.)

$$\int_1^{\infty} \ln x \cdot \frac{1}{x^2} dx = \left[\frac{-\ln x}{x} \right]_1^{\infty} - \int_1^{\infty} \frac{-1}{x^2} dx$$

$$u = \ln x \quad dv = \frac{1}{x^2} dx \quad = \left[\frac{-\ln x}{x} - \frac{1}{x} \right]_1^{\infty}$$

$$du = \frac{1}{x} dx \quad v = -\frac{1}{x}$$

$$= \left[\frac{-\ln x - 1}{x} \right]_1^{\infty} = \lim_{n \rightarrow \infty} \left(\frac{-\ln n - 1}{n} \right) - \left(\frac{-1}{1} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{-\ln n - 1}{n} + 1 = \frac{-\infty + 1}{\infty} + 1 = \lim_{n \rightarrow \infty} \frac{-1}{n} = 0 + 1 = 1$$

indeterminant ... use l'Hopital's

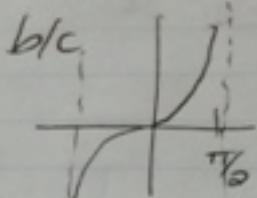
so b_n converges...

thus $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

Also converges b/c it's smaller than b_n .

$$(29) \sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$$

note that $\lim_{n \rightarrow \infty} \tan^{-1}(n) = \frac{\pi}{2}$



thus $\tan^{-1}(n) \leq \frac{\pi}{2}$

so $\frac{\tan^{-1}(n)}{n\sqrt{n}} \leq \frac{\pi/2}{n\sqrt{n}} = \frac{\pi/2}{n^{3/2}}$

$\tan \theta = \infty$ when $\theta = \frac{\pi}{2}$
(see graph)

so $\frac{\tan^{-1}(n)}{n\sqrt{n}} \leq \frac{\pi}{2} \left(\frac{1}{n^{3/2}} \right)$

p-series
w/ $p = 3/2$
converges

$$\sum_{n=1}^{\infty} \frac{\pi}{2} \left(\frac{1}{n^{3/2}} \right) = \frac{\pi}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n^{3/2}} \right) = \text{convergent}$$

since $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$ is smaller

than a convergent series,

it also converges

(by the comparison test)

$$(31) \sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$$

try the n^{th} root test...

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{5^n}{3^n + 4^n}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{5^n}}{\sqrt[n]{3^n + 4^n}} = \lim_{n \rightarrow \infty} \frac{5}{\sqrt[n]{3^n + 4^n}}$$

(NOTE THAT YOU CAN'T CANCEL THE POWERS ON 3^n AND 4^n b/c they are a sum)

let's look at what $\sqrt[n]{3^n + 4^n}$

does as n gets bigger. -

$$n=2: \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

$$n=3: \sqrt[3]{27 + 64} = \sqrt[3]{91} < 5$$

so it's getting smaller...

meaning that since denom is getting smaller...

$$\lim_{n \rightarrow \infty} \frac{5}{\sqrt[n]{3^n + 4^n}} = \infty > 1$$

so **DIVERGES**

$$(33) \sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{\sqrt{n}}$$

try limit comparison test
(go for when you have a fxn inside a trig fxn)

$$b_n = \left(\frac{\frac{1}{n}}{\sqrt{n}} \right)$$

$$\lim_{n \rightarrow \infty} \frac{\frac{\sin(\frac{1}{n})}{\sqrt{n}}}{\left(\frac{\frac{1}{n}}{\sqrt{n}} \right)} = \lim_{n \rightarrow \infty} \left(\frac{\sin \frac{1}{n}}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\frac{1}{n}} \right) = \frac{\sin 0}{0} = \frac{0}{0}$$

indeterminant
so use L'Hopital's

see next page

(33) (cont.)

$$\lim_{n \rightarrow \infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\cos(\frac{1}{n}) \cdot \frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n \rightarrow \infty} \cos(\frac{1}{n}) = \cos(0) = 1 = c > 0$$

So b_n and a_n do the same thing... let's figure out what b_n does

$$b_n = \frac{\frac{1}{n}}{\sqrt{n}} = \frac{1}{n} \cdot \frac{1}{\sqrt{n}} = \frac{1}{n^{3/2}} \leftarrow \begin{array}{l} \text{it's} \\ \text{a} \\ \text{p-series} \\ \text{p} = 3/2 \end{array}$$

So it converges

thus $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ also

converges