Polar Coordinate System

- Origin O is called the *pole*.
- A ray (usually) drawn in the direction of the positive x-axis in Cartesian coordinates is called the *polar* axis.
- A point P in the plane is identified by the coordinates (r, θ) where r is the distance of the point from the origin and θ is the angle between the polar axis and line OP measured counter-clockwise in radians.

Converting Cartesian and Polar

- To find Cartesian coordinates (x, y) when polar coordinates (r, θ) are known:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Note: This allows any function in the form $r = f(\theta)$ to be expressed parametrically as:

$$x = f(\theta)\cos\theta \qquad y = f(\theta)\sin\theta$$

- To find polar coordinates (r, θ) when Cartesian coordinates (x, y) are known:

$$r^2 = x^2 + y^2 \qquad \theta = \tan^{-1}(y/x)$$

Calculus with Polar Curves

- We wish to be able to do the following:
  - Take derivatives of polar curves
  - Find tangent lines.
  - Determine areas enclosed by polar curves.
  - Find the length of arcs from polar functions.

Derivatives of Polar Curves

- Because we can define polar curves in the form $r = f(\theta)$ parametrically with:

$$x = r\cos\theta \qquad y = r\sin\theta$$

- We can use our knowledge of derivatives of parametric functions (along with the product rule) for the polar case.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$

Derivatives of Polar Curves

- For the cardioid $r = 1 + \sin\theta$ find the slope of the tangent line when $\theta = \pi/3$.

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$$x = r\cos\theta = (1 + \sin\theta)\cos\theta$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 + \sin\theta)\cos\theta + \sin\theta\cos\theta}{(1 + \sin\theta)(-\sin\theta) + \cos^2\theta}$$

$$\frac{\cos\theta\,(1 + 2\sin\theta)}{\cos^2\theta - \sin\theta - \sin^2\theta} = \frac{\frac{1}{2}(1 + \sqrt{3})}{\frac{1}{4} - \frac{\sqrt{3}}{2} - \frac{3}{4}} = \frac{\frac{1}{2}(1 + \sqrt{3})}{-\frac{1}{2}(1 + \sqrt{3})} = -1$$

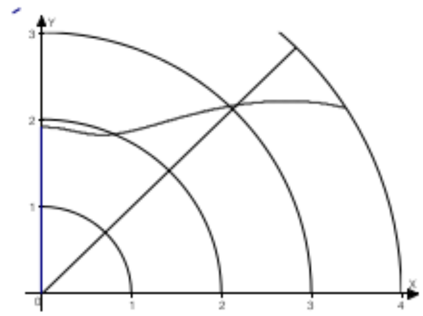
Derivatives of Polar Curves (calculator)

- Consider the three-petaled rose curve:

$$r = 2\sin 3\theta$$

1. Graph the curve using technology.
2. Find the (x, y) coordinates of the point where $\theta = \frac{\pi}{6}$.
3. Determine $\frac{dy}{dx}\big|_{\theta=\pi/6}$ for this polar curve.
4. Write the equation the line tangent to the given curve at $\theta = \pi/6$.

Area of Polar Curve



Areas of Polar Curves

- Area of a polar curve is therefore:

$$\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta$$

- A confounding issue with these areas is determining the limits of integration. It isn't always easy to visualize whether the limits should be $[0,2\pi]$ or something else, because there are both positive and negative r values.

Areas of Polar Curves

- Consider these two curves:

$$r = \cos 4\theta \qquad r = \cos 3\theta$$

- What shape are each of these?
- If we are finding the area of each of these curves, what are the appropriate limits of integration. Why?

Polar Arc Length

- Polar arc length is found by parameterizing the polar curve $(x = r\cos\theta\,, y = r\sin\theta)$ then finding $dx/d\theta$ and $dy/d\theta$ and using the parametric arc length formula we already know.

Polar Practice (Worksheet 4)

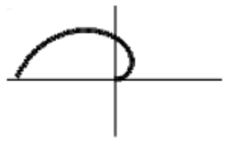
Work the following on notebook paper. Do not use your calculator except on problem 10.

1. Find the slope of the curve $r = 2 - 3\sin\theta$ at the point $(2, \pi)$.

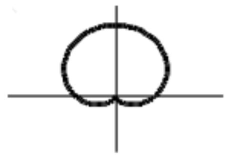
2. Find the equation of the tangent line to the curve $r = 3\sin(2\theta)$ at the point where $\theta = \dfrac{\pi}{3}$.

On problems 3 – 5, set up an integral to find the area of the shaded region. Do not evaluate.

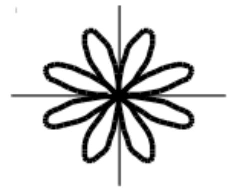
3. $r = \theta$



4. $r = 1 + \sin\theta$



5. $r = 2\sin 4\theta$



6. Sketch the polar region described by the following integral expression for area:

$$\frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\, d\theta$$

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7. (a) In polar coordinates, write equations for the line $x = 1$ and the circle of radius 2 centered at the origin.
   (b) Write the integral in polar coordinates representing the area of the region to the right of $x = 1$ and inside the circle.
   (c) Evaluate the integral.

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8. (a) Sketch the bounded region inside the lemniscate $r^2 = 4\cos(2\theta)$ and outside the circle $r = \sqrt{2}$.
   (b) Compute the area of the region described in part (a).

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9. Find the area between the two spirals $r = \theta$ and $r = 2\theta$ for $0 \le \theta \le 2\pi$.

Use your calculator on problem 10.

10. Given the polar curve $r = \theta + 2\sin\theta$ for $0 \le \theta \le 2\pi$
    (a) Sketch the graph of the curve.
    (b) Find the angle θ that corresponds to the point(s) on the curve where $x = -1$.
    (c) Find the angle θ that corresponds to the point(s) on the curve where $y = 2$.

Polar Practice (Worksheet 4)

Answers to Worksheet 4 on Polar

1. $\frac{2}{3}$
2. $y-\frac{9}{4}=\frac{\sqrt{3}}{5}\left(x-\frac{3\sqrt{3}}{4}\right)$
3. $\frac{1}{2}\int_0^{\pi}\theta^2\,d\theta$
4. $\frac{1}{2}\int_{\pi/2}^{\pi}(1+\sin\theta)^2\,d\theta$
5. $\frac{1}{2}\int_0^{\pi/4}(2\sin 4\theta)^2\,d\theta$
6. 1 petal of $r=\sin 3\theta$
7. (a) $r=2\sec\theta,\ r=2$

   (b) $\frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(2^2-\sec^2\theta\right)d\theta$

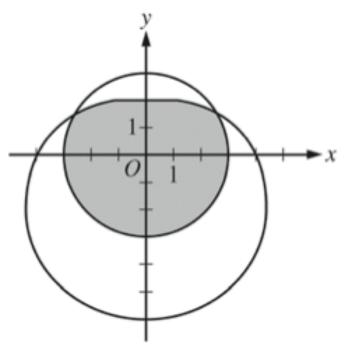
   (c) $\frac{4\pi-3\sqrt{3}}{3}$
8. (a) graph

   (b) $2\sqrt{3}-\frac{2\pi}{3}$
9. $4\pi^3$
10. (a) graph

    (b) 1.839, 4.295

    (c) 0.921, 2.563

A graphing calculator may be required for some parts of this problem.



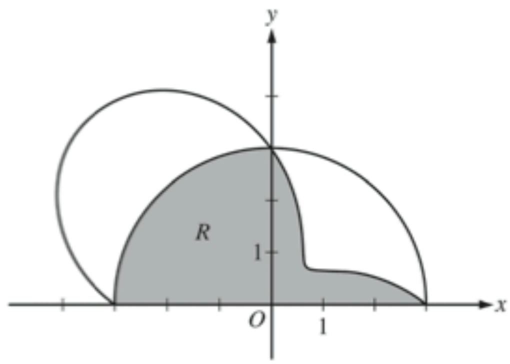
2. The graphs of the polar curves $r = 3$ and $r = 4 - 2\sin\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

   (a) Let S be the shaded region that is inside the graph of $r = 3$ and also inside the graph of $r = 4 - 2\sin\theta$. Find the area of S.

   (b) A particle moves along the polar curve $r = 4 - 2\sin\theta$ so that at time t seconds, $\theta = t^2$. Find the time t in the interval $1 \le t \le 2$ for which the x-coordinate of the particle's position is -1.

   (c) For the particle described in part (b), find the position vector in terms of t. Find the velocity vector at time $t = 1.5$.

A graphing calculator may be required for some parts of this problem.



2. The graphs of the polar curves $r = 3$ and $r = 3 - 2\sin(2\theta)$ are shown in the figure above for $0 \le \theta \le \pi$.

(a) Let R be the shaded region that is inside the graph of $r = 3$ and inside the graph of $r = 3 - 2\sin(2\theta)$. Find the area of R.

(b) For the curve $r = 3 - 2\sin(2\theta)$, find the value of $\frac{dx}{d\theta}$ at $\theta = \frac{\pi}{6}$.

(c) The distance between the two curves changes for $0 < \theta < \frac{\pi}{2}$. Find the rate at which the distance between the two curves is changing with respect to θ when $\theta = \frac{\pi}{3}$.

(d) A particle is moving along the curve $r = 3 - 2\sin(2\theta)$ so that $\frac{d\theta}{dt} = 3$ for all times $t \ge 0$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{6}$.