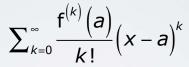
# SOLVED PROBLEMS ON TAYLOR AND MACLAURIN SERIES

# TAYLOR AND MACLAURIN SERIES

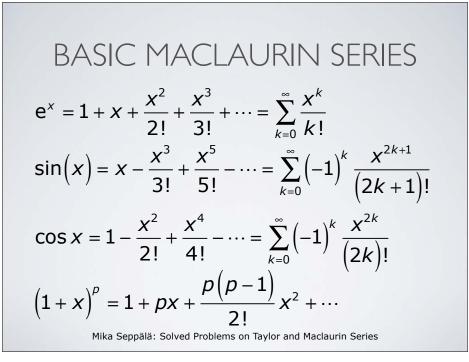
Taylor Series of a function f at x = a is



It is a Power Series centered at a.

Maclaurin Series of a function f is a Taylor Series at x = 0.

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# USE TAYLOR SERIES

- To estimate values of functions on an interval.
- 2 To compute limits of functions.
- 3 To approximate integrals.
- 4 To study properties of the function in question.

# FINDING TAYLOR SERIES

To find Taylor series of functions, we may:

- 1 Use substitution.
- 2 Differentiate known series term by term.
- 3 Integrate known series term by term.
- 4 Add, divide, and multiply known series.

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#### OVERVIEW OF PROBLEMS Find the Maclaurin Series of the following functions. sin(x)2 $\sin(x^2)$ $\arctan(x)$ 3 1 $\cos^2(x)$ 5 6 $x^2 e^x$ 4 $x^2 \arctan(x^3)$ 9 8 7 $\sinh(x)$ Mika Seppälä: Solved Problems on Taylor and Maclaurin Series

# OVERVIEW OF PROBLEMS

Find the Taylor Series of the following functions at the given value of *a*.

10 
$$x - x^{3}$$
 at  $a = -2$  11  $\frac{1}{x}$  at  $a = 2$   
12  $e^{-2x}$  at  $a = 1/2$  13  $\sin(x)$  at  $a = \pi/4$   
14  $10^{x}$  at  $a = 1$  15  $\ln(1+x)$  at  $a = -2$ 

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Find the Maclaurin Series of the following functions.

General Formula for finding the Taylor Polynomial of a Function

**6** 
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$
  
=  $f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \cdots$ 

The series in Equation 6 is called the **Taylor series of the function** f at a (or about a or centered at a). For the special case a = 0 the Taylor series becomes

Practice: (Complete problems #10-15 for the "Taylor/Maclaurin HW") Find the Taylor Series of the following functions at the given value of *a*. 10  $x - x^3$  at a = -2 11  $\frac{1}{x}$  at a = 212  $e^{-2x}$  at a = 1/2 13  $\sin(x)$  at  $a = \pi/4$ 14  $10^x$  at a = 1 15  $\ln(1+x)$  at a = -2

7 
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots$$

This case arises frequently enough that it is given the special name Maclaurin series.

There are several MacLaurin Series that are SO common, they are worth memorizing. You should memorize all of these and the radius of convergence for each.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots$$
  $R = \infty$ 

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^{k} = \sum_{n=0}^{\infty} \binom{k}{n} x^{n} = 1 + kx + \frac{k(k-1)}{2!} x^{2} + \frac{k(k-1)(k-2)}{3!} x^{3} + \cdots \quad R = 1$$

Practice: (Complete problems #1-9 for the "Taylor/Maclaurin HW") Find the Maclaurin Series of the following functions.

1 
$$\sin(x^2)$$
 2  $\frac{\sin(x)}{x}$  3  $\arctan(x)$   
4  $\cos^2(x)$  5  $x^2e^x$  6  $\sqrt{1-x^3}$   
7  $\sinh(x)$  8  $\frac{e^x}{1-x}$  9  $x^2\arctan(x^3)$ 

Practice AP BC Problems:

#### 2014 AP<sup>®</sup> CALCULUS BC FREE-RESPONSE QUESTIONS

- 6. The Taylor series for a function f about x = 1 is given by  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n$  and converges to f(x) for
  - |x-1| < R, where R is the radius of convergence of the Taylor series.
  - (a) Find the value of R.
  - (b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about x = 1.
  - (c) The Taylor series for f' about x = 1, found in part (b), is a geometric series. Find the function f' to which the series converges for |x-1| < R. Use this function to determine f for |x-1| < R.

#### 2013 AP° CALCULUS BC FREE-RESPONSE QUESTIONS

- 6. A function f has derivatives of all orders at x = 0. Let  $P_n(x)$  denote the *n*th-degree Taylor polynomial for f about x = 0.
  - (a) It is known that f(0) = -4 and that  $P_1\left(\frac{1}{2}\right) = -3$ . Show that f'(0) = 2.
  - (b) It is known that  $f''(0) = -\frac{2}{3}$  and  $f'''(0) = \frac{1}{3}$ . Find  $P_3(x)$ .
  - (c) The function h has first derivative given by h'(x) = f(2x). It is known that h(0) = 7. Find the third-degree Taylor polynomial for h about x = 0.

#### **2005 AP CALCULUS BC FREE-RESPONSE QUESTIONS**

6. Let f be a function with derivatives of all orders and for which f(2) = 7. When n is odd, the nth derivative

of f at x = 2 is 0. When n is even and  $n \ge 2$ , the nth derivative of f at x = 2 is given by  $f^{(n)}(2) = \frac{(n-1)!}{3^n}$ .

- (a) Write the sixth-degree Taylor polynomial for f about x = 2.
- (b) In the Taylor series for f about x = 2, what is the coefficient of  $(x 2)^{2n}$  for  $n \ge 1$ ?
- (c) Find the interval of convergence of the Taylor series for f about x = 2. Show the work that leads to your answer.

#### \*\*SOME PARTS OF THIS PROBLEM MAY REQUIRE A GRAPHING CALCULATOR\*\* 2005 AP<sup>®</sup> CALCULUS BC FREE-RESPONSE QUESTIONS (Form B)

3. The Taylor series about x = 0 for a certain function f converges to f(x) for all x in the interval of convergence. The nth derivative of f at x = 0 is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n (n-1)^2} \text{ for } n \ge 2$$

The graph of f has a horizontal tangent line at x = 0, and f(0) = 6.

- (a) Determine whether f has a relative maximum, a relative minimum, or neither at x = 0. Justify your answer.
- (b) Write the third-degree Taylor polynomial for f about x = 0.
- (c) Find the radius of convergence of the Taylor series for f about x = 0. Show the work that leads to your answer.

MACLAURIN SERIESProblem 1
$$f(x) = sin(x^2)$$
SolutionSubstitute x by  $x^2$  in the Maclaurin Series of sine.Hence  $sin(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2^{k+1}}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}$ Mixa Sepsili: Solved Problems on Taylor and Maclaurin Series

$$\begin{aligned} \text{DACLAURIN SERIES} \\ \hline \text{olution(cont'd)} \\ \text{ence } f'(x) &= \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}. \end{aligned}$$

$$\text{y integrating both sides, we obtain} \\ f'(x) &= \int \left(\sum_{k=0}^{\infty} (-1)^k x^{2k}\right) dx = \sum_{k=0}^{\infty} (-1)^k \int x^{2k} dx \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + C. \end{aligned}$$

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MACLAURIN SERIES

Problem 3

 $f(x) = \arctan(x)$ 

#### Solution

Observe that  $f'(x) = \frac{1}{1+x^2}$ . To find the Maclaurin Series of f'(x) substitute  $-x^2$  for x in Basic Power Series formula.

# MACLAURIN SERIES

#### Solution(cont'd)

0 is in the interval of convergence. Therefore we can insert x = 0 to find that the integration constant c = 0. Hence the Maclaurin series of  $\operatorname{arctan}(x)$  is

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

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### MACLAURIN SERIES

Problem 4

$$f(x) = \cos^2(x)$$

#### Solution

By the trigonometric identity,  $\cos^{2}(x) = (1 + \cos(2x))/2.$ 

Therefore we start with the Maclaurin Series of cosine.

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MACLAURIN SERIES  
Solution(cont'd)  
Substitute x by 2x in 
$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$
  
Thus  $\cos(2x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!}$ . After adding 1  
and dividing by 2, we obtain

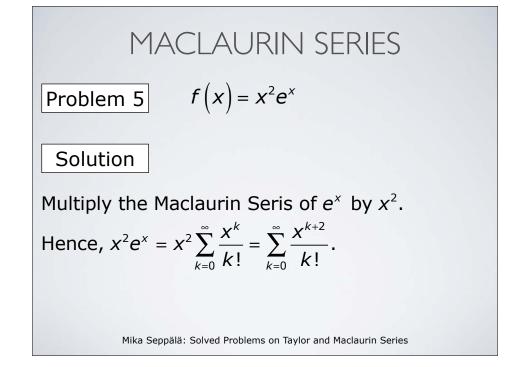
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$$\frac{\text{MACLAURIN SERIES}}{\text{Solution}(\text{cont'd})}$$

$$\cos^{2}(x) = \frac{1}{2} \left( 1 + \sum_{k=0}^{\infty} \left( -1 \right)^{k} \frac{\left( 2x \right)^{2k}}{\left( 2k \right)!} \right)$$

$$= \frac{1}{2} \left( 1 + 1 - \frac{\left( 2x \right)^{2}}{2!} + \frac{\left( 2x \right)^{4}}{4!} - \dots \right)$$

$$= 1 + \sum_{k=1}^{\infty} \left( -1 \right)^{k} \frac{2^{2k-1}}{\left( 2k \right)!} x^{2k}$$



Problem 6

$$f(x) = \sqrt{1-x^3}$$

#### Solution

By rewriting  $f(x) = (1 + (-x^3))^{1/2}$ . By substituting xby  $-x^3$  in the binomial formula with p = 1/2we obtain ,

 $\sqrt{1-x^3} = 1 - \frac{1}{2}x^3 - \frac{1}{8}x^6 - \dots$ 

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MACLAURIN SERIES Problem 7  $f(x) = \sinh(x)$ Solution By rewriting  $f(x) = \frac{e^x - e^{-x}}{2}$ . Substitute x by -x in the Maclaurin Series of  $e^x = 1 + x + \frac{x^2}{2} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ ,

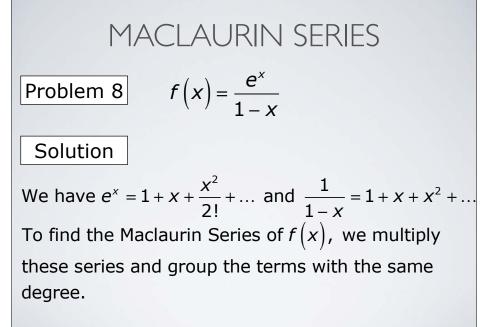
# MACLAURIN SERIES

Solution(cont'd)

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 - x + \frac{x^2}{2!} - \cdots$$

Thus when we add  $e^x$  and  $e^{-x}$ , the terms with odd power are canceled and the terms with even power are doubled. After dividing by 2, we obtain

$$\sinh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$



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# MACLAURIN SERIES

# Solution(cont'd) $\left(1 + x + \frac{x^2}{2!} + \dots\right) \times \left(1 + x + x^2 + \dots\right)$ $= 1 + 2x + \left(1 + 1 + \frac{1}{2!}\right)x^2 + \text{higher degree terms}$ $= 1 + 2x + \frac{5}{2}x^2 + \text{higher degree terms}$ Mika Sepälä: Solved Problems on Taylor and Maclaurin Series

MACLAURIN SERIES

Problem 9

 $f(x) = x^2 \arctan(x^3)$ 

#### Solution

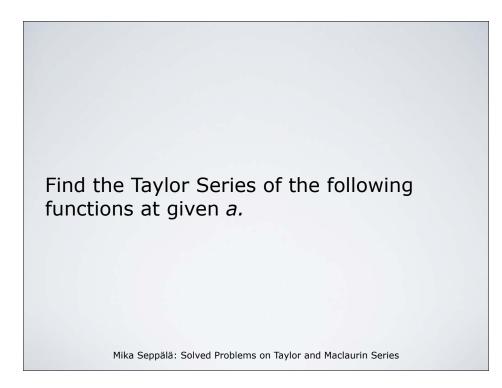
We have calculated the Maclaurin Series of  $\arctan(x)$ 

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

Substituting x by  $x^3$  in the above formula, we obtain

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$$\begin{aligned} \text{MACLAURIN SERIES} \\ \hline \text{Solution(cont'd)} \\ & \arctan\left(x^3\right) = \sum_{k=0}^{\infty} \left(-1\right)^k \frac{\left(x^3\right)^{2^{k+1}}}{2^{k+1}} = \sum_{k=0}^{\infty} \left(-1\right)^k \frac{x^{6^{k+3}}}{2^{k+1}}. \end{aligned} \\ \hline \text{Multiplying by } x^2 \text{ gives the desired Maclaurin Series} \\ & x^2 \arctan\left(x^3\right) = x^2 \sum_{k=0}^{\infty} \left(-1\right)^k \frac{x^{6^{k+3}}}{2^{k+1}} = \sum_{k=0}^{\infty} \left(-1\right)^k \frac{x^{6^{k+4}}}{2^{k+1}}. \end{aligned}$$



TAYLOR SERIESProblem 10
$$f(x) = x - x^3$$
 at  $a = -2$ SolutionTaylor Series of  $f(x) = x - x^3$  at  $a = -2$  is of the form $f(-2) + f^{(1)}(-2)(x+2) + \frac{f^{(2)}(-2)}{2!}(x+2)^2$  $+ \frac{f^{(3)}(-2)}{3!}(x+2)^3 + \frac{f^{(4)}(-2)}{4!}(x+2)^4 + \dots$ Mika Seppälä: Solved Problems on Taylor and Maclaurin Series

#### Solution(cont'd)

Since *f* is a polynominal function of degree 3, its derivatives of order higher than 3 is 0. Thus Taylor Series is of the form

$$f(-2) + f^{(1)}(-2)(x+2) + \frac{f^{(2)}(-2)}{2!}(x+2)^2 + \frac{f^{(3)}(-2)}{3!}(x+2)^3$$

# TAYLOR SERIES

#### Solution(cont'd)

By direct computation, f(-2) = 6,  $f^{(1)}(-2) = -11$ ,  $f^{(2)}(-2) = 12$ ,  $f^{(3)}(-2) = -6$ So the Taylor Series of  $x - x^3$  at a = -2 is  $6 - 11(x + 2) + 6(x + 2)^2 - (x + 2)^3$ 

TAYLOR SERIESProblem 11
$$f(x) = \frac{1}{x}$$
 at  $a = 2$ SolutionTaylor Series of  $f(x) = 1/x$  at  $a = 2$  is of the form $\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!} (x-2)^k$ . We need to find the generalexpression of the  $k^{th}$  derivative of  $1/x$ .

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# TAYLOR SERIES

#### Solution(cont'd)

We derive 1/x until a pattern is found.  $f(x) = 1/x = x^{-1}, f^{(1)}(x) = (-1)x^{-2}$   $f^{(2)}(x) = (-1)(-2)x^{-3}, f^{(3)}(x) = (-1)(-2)(-3)x^{-4}$ In general,  $f^{(k)}(x) = (-1)^{k} k! x^{-(k+1)}$ . Therefore  $f^{(k)}(2) = (-1)^{k} k! 2^{-(k+1)}$ .

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# TAYLOR SERIES

#### Solution(cont'd)

(.)

After inserting the general expression of the  $k^{\text{th}}$  derivative evaluated at 2 we obtain,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!} (x-2)^{k} = \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^{k} k! 2^{-(k+1)} (x-2)^{k}$$

Hence, the the Taylor Series of  $\frac{1}{x}$  is  $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{(k+1)}} (x-2)^k$ .

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TAYLOR SERIESProblem 12
$$f(x) = e^{-2x}$$
 at  $a = 1/2$ SolutionTaylor Series of  $f(x) = e^{-2x}$  at  $a = 1/2$  is of the form $\sum_{k=0}^{\infty} \frac{f^{(k)}(1/2)}{k!} \left(x - \frac{1}{2}\right)^k$ . We need to find the generalexpression of the  $k^{th}$  derivative of  $e^{-2x}$ .

#### Solution(cont'd)

We derive  $e^{-2x}$  until a pattern is found.  $f(x) = e^{-2x}, f^{(1)}(x) = -2e^{-2x}, f^{(2)}(x) = -2 - 2e^{-2x}$ In general,  $f^{(k)}(x) = (-1)^{k} 2^{k} e^{-2x}$ . Therefore  $f^{(k)}(1/2) = (-1)^{k} 2^{k} e^{-2x\frac{1}{2}} = \frac{(-1)^{k} 2^{k}}{e}$ .

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# TAYLOR SERIES

#### Solution(cont'd)

After inserting the general expression of the  $k^{\text{th}}$  derivative evaluated at 1/2 we obtain,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1/2)}{k!} \left(x - \frac{1}{2}\right)^{k} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\left(-1\right)^{k} 2^{k}}{e} \left(x - \frac{1}{2}\right)^{k}$$

Hence, the the Taylor Series of  $e^{-2x}$  is

$$\sum_{k=0}^{\infty} \frac{\left(-1\right)^{k}}{e \times \left(k!\right)} \left(2x-1\right)^{k}.$$

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TAYLOR SERIESProblem 13f(x) = sin(x) at  $a = \pi/4$ SolutionTaylor Series of f(x) = sin(x) at  $a = \pi/4$  is of theform  $\sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/4)}{k!} \left(x - \frac{\pi}{4}\right)^k$ . We need to find thegeneral expression of the  $k^{th}$  derivative of sin(x).

# TAYLOR SERIES

#### Solution(cont'd)

We derive  $\sin(x)$  until a pattern is found.  $f(x) = \sin(x), f^{(1)}(x) = \cos(x), f^{(2)}(x) = -\sin(x)$ In general,  $f^{(k)}(x) = \begin{cases} \sin(x) & \text{if } k = 4n \\ \cos(x) & \text{if } k = 4n+1 \\ -\sin(x) & \text{if } k = 4n+2 \\ -\cos(x) & \text{if } k = 4n+3 \end{cases}$ 

#### Solution(cont'd)

In other words, even order derivatives are either sin(x)or -sin(x) and odd order derivatives are either cos(x)or -cos(x). So the Taylor Series at  $a = \pi/4$  can be written as

$$\sum_{k=0}^{\infty} \left(-1\right)^{k} \frac{\sin\left(\frac{\pi}{4}\right)}{\left(2k\right)!} \left(x - \frac{\pi}{4}\right)^{2k} + \sum_{k=0}^{\infty} \left(-1\right)^{k} \frac{\cos\left(\frac{\pi}{4}\right)}{\left(2k + 1\right)!} \left(x - \frac{\pi}{4}\right)^{2k+2k}$$

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# TAYLOR SERIES

#### Solution(cont'd)

Since, at  $a = \pi/4$ ,  $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$ , the Taylor Series can be simplified to

$$\sum_{k=0}^{\infty} \frac{\left(-1\right)^{k}}{\sqrt{2}\left(2k\right)!} \left(x - \frac{\pi}{4}\right)^{2k} + \sum_{k=0}^{\infty} \frac{\left(-1\right)^{k}}{\sqrt{2}\left(2k + 1\right)!} \left(x - \frac{\pi}{4}\right)^{2k+1}$$

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TAYLOR SERIES

$$f(x) = 10^{x}$$
 at  $a = 10^{x}$ 

Solution

Taylor Series of  $f(x) = 10^x$  at a = 1 is of the form  $\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (x-1)^k$ . We need to find the general expression of the  $k^{\text{th}}$  derivative of  $10^x$ .

# TAYLOR SERIES

#### Solution(cont'd)

We derive  $10^x$  until a pattern is found.  $f(x) = 10^x$ ,  $f^{(1)}(x) = \ln(10) \times 10^x$ ,  $f^{(2)}(x) = \ln^2(10)10^x$ In general,  $f^{(k)}(x) = \ln^k(10)10^x$ . Therefore  $f^{(k)}(1) = \ln^k(10)10$ .

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#### Solution(cont'd)

After inserting the general expression of the  $k^{\text{th}}$  derivative evaluated at 1 we obtain,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (x-1)^{k} = \sum_{k=0}^{\infty} \frac{\ln^{k}(10)10}{k!} (x-1)^{k}$$

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# TAYLOR SERIES Problem 15 $f(x) = \ln(1+x)$ at a = -2

#### Solution

Taylor Series of  $f(x) = \ln(1+x)$  at a = -2 is of the form  $\sum_{k=0}^{\infty} \frac{f^{(k)}(-2)}{k!} (x+2)^k$ . We need to find the general expression of the  $k^{\text{th}}$  derivative of  $\ln(1+x)$ .

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# TAYLOR SERIESSolution(cont'd)We derive ln(x+1) until a pattern is found. $f(x) = ln(x+1), f^{(1)}(x) = \frac{1}{x+1}, f^{(2)}(x) = -\frac{1}{(x+1)^2}$ In general, $f^{(k)}(x) = \frac{(-1)^k}{(x+1)^k}$ . Therefore $f^{(k)}(-2) = 1$ .

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# TAYLOR SERIES

#### Solution(cont'd)

After inserting the general expression of the  $k^{\text{th}}$  derivative evaluated at -2 we obtain

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(-2)}{k!} (x+2)^k = \sum_{k=0}^{\infty} \frac{1}{k!} (x+2)^k.$$