SOLVED PROBLEMS ON TAYLOR AND MACLAURIN SERIES

TAYLOR AND MACLAURIN SERIES

Taylor Series of a function f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

It is a Power Series centered at a.

Maclaurin Series of a function f is a Taylor Series at $x = 0$.

BASIC MACLAURIN SERIES

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots$$

USE TAYLOR SERIES

$\boxed{1}$ To estimate values of functions on an interval.

$\boxed{2}$ To compute limits of functions.

$\boxed{3}$ To approximate integrals.

$\boxed{4}$ To study properties of the function in question.

FINDING TAYLOR SERIES

To find Taylor series of functions, we may:

1. Use substitution.
2. Differentiate known series term by term.
3. Integrate known series term by term.
4. Add, divide, and multiply known series.

OVERVIEW OF PROBLEMS

Find the Maclaurin Series of the following functions.

1. $\sin\left(x^2\right)$
2. $\dfrac{\sin(x)}{x}$
3. $\arctan(x)$
4. $\cos^2(x)$
5. $x^2 e^x$
6. $\sqrt{1-x^3}$
7. $\sinh(x)$
8. $\dfrac{e^x}{1-x}$
9. $x^2 \arctan\left(x^3\right)$

OVERVIEW OF PROBLEMS

Find the Taylor Series of the following functions at the given value of *a*.

10. $x - x^3$ at $a = -2$
11. $\dfrac{1}{x}$ at $a = 2$
12. e^{-2x} at $a = 1/2$
13. $\sin(x)$ at $a = \pi/4$
14. 10^x at $a = 1$
15. $\ln(1+x)$ at $a = -2$

Find the Maclaurin Series of the following functions.

Taylor and MacLaurin Series Unit

General Formula for finding the Taylor Polynomial of a Function

$$\boxed{6}\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***). For the special case $a = 0$ the Taylor series becomes

Practice: (Complete problems #10-15 for the "Taylor/Maclaurin HW")

Find the Taylor Series of the following functions at the given value of *a*.

$\boxed{10}$ $x - x^3$ at $a = -2$

$\boxed{11}$ $\dfrac{1}{x}$ at $a = 2$

$\boxed{12}$ e^{-2x} at $a = 1/2$

$\boxed{13}$ $\sin(x)$ at $a = \pi/4$

$\boxed{14}$ 10^x at $a = 1$

$\boxed{15}$ $\ln(1+x)$ at $a = -2$

$$\boxed{7}\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

There are several MacLaurin Series that are SO common, they are worth memorizing. You should memorize all of these and the radius of convergence for each.

Taylor and MacLaurin Series Unit

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Practice: (Complete problems #1-9 for the "Taylor/Maclaurin HW")

Find the Maclaurin Series of the following functions.

1	$\sin(x^2)$	2	$\dfrac{\sin(x)}{x}$	3	$\arctan(x)$
4	$\cos^2(x)$	5	$x^2 e^x$	6	$\sqrt{1-x^3}$
7	$\sinh(x)$	8	$\dfrac{e^x}{1-x}$	9	$x^2 \arctan(x^3)$

Practice AP BC Problems:

Taylor and MacLaurin Series Unit

2014 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n$ and converges to $f(x)$ for $|x - 1| < R$, where R is the radius of convergence of the Taylor series.

   (a) Find the value of R.

   (b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

   (c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x - 1| < R$. Use this function to determine f for $|x - 1| < R$.

Taylor and MacLaurin Series Unit

2013 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

   (a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

   (b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

   (c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

Taylor and MacLaurin Series Unit

2005 AP CALCULUS BC FREE-RESPONSE QUESTIONS

6. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

   (a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

   (b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

   (c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

Taylor and MacLaurin Series Unit

SOME PARTS OF THIS PROBLEM MAY REQUIRE A GRAPHING CALCULATOR

2005 AP® CALCULUS BC FREE-RESPONSE QUESTIONS (Form B)

3. The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2} \text{ for } n \geq 2.$$

The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

MACLAURIN SERIES

Problem 1 $f(x) = \sin(x^2)$

Solution

Substitute x by x^2 in the Maclaurin Series of sine.

Hence $\sin(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}$

MACLAURIN SERIES

Problem 2 $f(x) = \frac{\sin(x)}{x}$

Solution

Divide the Maclaurin Series of sine by x. Hence,

$$\frac{\sin(x)}{x} = \frac{1}{x} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}$$

MACLAURIN SERIES

Problem 3 $f(x) = \arctan(x)$

Solution

Observe that $f'(x) = \frac{1}{1+x^2}$. To find the Maclaurin Series of $f'(x)$ substitute $-x^2$ for x in Basic Power Series formula.

MACLAURIN SERIES

Solution(cont'd)

Hence $f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k}$.

By integrating both sides, we obtain

$$f(x) = \int \left(\sum_{k=0}^{\infty} (-1)^k x^{2k} \right) dx = \sum_{k=0}^{\infty} (-1)^k \int x^{2k}\, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + C.$$

MACLAURIN SERIES

Solution(cont'd)

0 is in the interval of convergence. Therefore we can insert $x = 0$ to find that the integration constant $c = 0$. Hence the Maclaurin series of $\arctan(x)$ is

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

MACLAURIN SERIES

Problem 4 $\quad f(x) = \cos^2(x)$

Solution

By the trigonometric identity,
$\cos^2(x) = \left(1 + \cos(2x)\right)/2.$

Therefore we start with the Maclaurin Series of cosine.

MACLAURIN SERIES

Solution(cont'd)

Substitute x by $2x$ in $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.

Thus $\cos(2x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!}$. After adding 1 and dividing by 2, we obtain

MACLAURIN SERIES

Solution(cont'd)

$$\cos^2(x) = \frac{1}{2}\left(1 + \sum_{k=0}^{\infty} (-1)^k \frac{(2x)^{2k}}{(2k)!}\right)$$

$$= \frac{1}{2}\left(1 + 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \dots\right)$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k-1}}{(2k)!} x^{2k}$$

MACLAURIN SERIES

Problem 5 $f(x) = x^2 e^x$

Solution

Multiply the Maclaurin Seris of e^x by x^2.

Hence, $x^2 e^x = x^2 \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}$.

MACLAURIN SERIES

Problem 6 $f(x) = \sqrt{1 - x^3}$

Solution

By rewriting $f(x) = \left(1 + \left(-x^3\right)\right)^{1/2}$. By substituting x by -x^3 in the binomial formula with $p = 1/2$ we obtain ,

$$\sqrt{1 - x^3} = 1 - \frac{1}{2} x^3 - \frac{1}{8} x^6 - \ldots$$

MACLAURIN SERIES

Problem 7 $f(x) = \sinh(x)$

Solution

By rewriting $f(x) = \frac{e^x - e^{-x}}{2}$. Substitute x by $-x$ in the Maclaurin Series of $e^x = 1 + x + \frac{x^2}{2} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$,

MACLAURIN SERIES

Solution(cont'd)

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 - x + \frac{x^2}{2!} - \cdots$$

Thus when we add e^x and e^{-x}, the terms with odd power are canceled and the terms with even power are doubled. After dividing by 2, we obtain

$$\sinh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

MACLAURIN SERIES

Problem 8 $f(x) = \dfrac{e^x}{1-x}$

Solution

We have $e^x = 1 + x + \dfrac{x^2}{2!} + \dots$ and $\dfrac{1}{1-x} = 1 + x + x^2 + \dots$

To find the Maclaurin Series of $f(x)$, we multiply these series and group the terms with the same degree.

MACLAURIN SERIES

Solution(cont'd)

$$\left(1 + x + \frac{x^2}{2!} + \dots\right) \times \left(1 + x + x^2 + \dots\right)$$

$$= 1 + 2x + \left(1 + 1 + \frac{1}{2!}\right)x^2 + \text{higher degree terms}$$

$$= 1 + 2x + \frac{5}{2}x^2 + \text{higher degree terms}$$

MACLAURIN SERIES

Problem 9 $f(x) = x^2 \arctan(x^3)$

Solution

We have calculated the Maclaurin Series of $\arctan(x)$

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Substituting x by x^3 in the above formula, we obtain

MACLAURIN SERIES

Solution(cont'd)

$$\arctan(x^3) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(x^3\right)^{2k+1}}{2k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+3}}{2k+1}.$$

Multiplying by x^2 gives the desired Maclaurin Series

$$x^2 \arctan(x^3) = x^2 \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+3}}{2k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{6k+4}}{2k+1}$$

Find the Taylor Series of the following functions at given *a*.

TAYLOR SERIES

Problem 10 $f(x) = x - x^3$ at $a = -2$

Solution

Taylor Series of $f(x) = x - x^3$ at $a = -2$ is of the form

$$f(-2) + f^{(1)}(-2)(x+2) + \frac{f^{(2)}(-2)}{2!}(x+2)^2 + \frac{f^{(3)}(-2)}{3!}(x+2)^3 + \frac{f^{(4)}(-2)}{4!}(x+2)^4 + \ldots$$

TAYLOR SERIES

Solution(cont'd)

Since f is a polynominal function of degree 3, its derivatives of order higher than 3 is 0. Thus Taylor Series is of the form

$$f(-2) + f^{(1)}(-2)(x+2) + \frac{f^{(2)}(-2)}{2!}(x+2)^2 + \frac{f^{(3)}(-2)}{3!}(x+2)^3$$

TAYLOR SERIES

Solution(cont'd)

By direct computation,

$$f(-2) = 6,\ f^{(1)}(-2) = -11,\ f^{(2)}(-2) = 12,\ f^{(3)}(-2) = -6$$

So the Taylor Series of $x - x^3$ at $a = -2$ is

$$6\text{-}11(x+2) + 6(x+2)^2 - (x+2)^3$$

TAYLOR SERIES

Problem 11 $f(x) = \dfrac{1}{x}$ at $a = 2$

Solution

Taylor Series of $f(x) = 1/x$ at $a = 2$ is of the form

$\sum_{k=0}^{\infty} \dfrac{f^{(k)}(2)}{k!}(x-2)^k$. We need to find the general expression of the k^{th} derivative of $1/x$.

TAYLOR SERIES

Solution(cont'd)

We derive $1/x$ until a pattern is found.

$f(x) = 1/x = x^{-1}$, $f^{(1)}(x) = (-1)x^{-2}$

$f^{(2)}(x) = (-1)(-2)x^{-3}$, $f^{(3)}(x) = (-1)(-2)(-3)x^{-4}$

In general, $f^{(k)}(x) = (-1)^k k!x^{-(k+1)}$. Therefore

$f^{(k)}(2) = (-1)^k k!2^{-(k+1)}$.

TAYLOR SERIES

Solution(cont'd)

After inserting the general expression of the k^{th} derivative evaluated at 2 we obtain,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k k!2^{-(k+1)}(x-2)^k$$

Hence, the the Taylor Series of $\dfrac{1}{x}$ is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{(k+1)}}(x-2)^k$.

TAYLOR SERIES

Problem 12 $f(x) = e^{-2x}$ at $a = 1/2$

Solution

Taylor Series of $f(x) = e^{-2x}$ at $a = 1/2$ is of the form

$\sum_{k=0}^{\infty} \dfrac{f^{(k)}(1/2)}{k!}\left(x - \dfrac{1}{2}\right)^k$. We need to find the general expression of the k^{th} derivative of e^{-2x}.

TAYLOR SERIES

Solution(cont'd)

We derive e^{-2x} until a pattern is found.

$$f(x) = e^{-2x},\ f^{(1)}(x) = -2e^{-2x},\ f^{(2)}(x) = -2 - 2e^{-2x}$$

In general, $f^{(k)}(x) = (-1)^k 2^k e^{-2x}$.

Therefore $f^{(k)}(1/2) = (-1)^k 2^k e^{-2\times\frac{1}{2}} = \dfrac{(-1)^k 2^k}{e}$.

TAYLOR SERIES

Solution(cont'd)

After inserting the general expression of the k^{th} derivative evaluated at $1/2$ we obtain,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1/2)}{k!}\left(x-\frac{1}{2}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{(-1)^k 2^k}{e}\left(x-\frac{1}{2}\right)^k$$

Hence, the the Taylor Series of e^{-2x} is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{e\times(k!)}(2x-1)^k.$$

TAYLOR SERIES

Problem 13 $f(x) = \sin(x)$ at $a = \pi/4$

Solution

Taylor Series of $f(x) = \sin(x)$ at $a = \pi/4$ is of the form $\sum_{k=0}^{\infty} \dfrac{f^{(k)}(\pi/4)}{k!}\left(x-\dfrac{\pi}{4}\right)^k$. We need to find the general expression of the k^{th} derivative of $\sin(x)$.

TAYLOR SERIES

Solution(cont'd)

We derive $\sin(x)$ until a pattern is found.

$$f(x) = \sin(x),\ f^{(1)}(x) = \cos(x),\ f^{(2)}(x) = -\sin(x)$$

In general, $f^{(k)}(x) = \begin{cases} \sin(x) & \text{if } k = 4n \\ \cos(x) & \text{if } k = 4n+1 \\ -\sin(x) & \text{if } k = 4n+2 \\ -\cos(x) & \text{if } k = 4n+3 \end{cases}$

TAYLOR SERIES

Solution(cont'd)

In other words, even order derivatives are either $\sin(x)$ or $-\sin(x)$ and odd order derivatives are either $\cos(x)$ or $-\cos(x)$. So the Taylor Series at $a=\pi/4$ can be written as

$$\sum_{k=0}^{\infty}(-1)^k\frac{\sin(\pi/4)}{(2k)!}\left(x-\frac{\pi}{4}\right)^{2k}+\sum_{k=0}^{\infty}(-1)^k\frac{\cos(\pi/4)}{(2k+1)!}\left(x-\frac{\pi}{4}\right)^{2k+1}$$

TAYLOR SERIES

Solution(cont'd)

Since, at $a=\pi/4$, $\sin(\pi/4)=\cos(\pi/4)=1/\sqrt{2}$, the Taylor Series can be simplified to

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2}(2k)!}\left(x-\frac{\pi}{4}\right)^{2k}+\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{2}(2k+1)!}\left(x-\frac{\pi}{4}\right)^{2k+1}.$$

TAYLOR SERIES

Problem 14 $\quad f(x)=10^x$ at $a=1$

Solution

Taylor Series of $f(x)=10^x$ at $a=1$ is of the form $\sum_{k=0}^{\infty}\frac{f^{(k)}(1)}{k!}(x-1)^k$. We need to find the general expression of the k^{th} derivative of 10^x.

TAYLOR SERIES

Solution(cont'd)

We derive 10^x until a pattern is found.

$f(x)=10^x$, $f^{(1)}(x)=\ln(10)\times 10^x$, $f^{(2)}(x)=\ln^2(10)10^x$

In general, $f^{(k)}(x)=\ln^k(10)10^x$.

Therefore $f^{(k)}(1)=\ln^k(10)10$.

TAYLOR SERIES

Solution(cont'd)

After inserting the general expression of the k^{th} derivative evaluated at 1 we obtain,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{\infty} \frac{\ln^k(10)10}{k!}(x-1)^k$$

TAYLOR SERIES

Problem 15 $f(x) = \ln(1+x)$ at $a = -2$

Solution

Taylor Series of $f(x) = \ln(1+x)$ at $a = -2$ is of the form $\sum_{k=0}^{\infty} \frac{f^{(k)}(-2)}{k!}(x+2)^k$. We need to find the general expression of the k^{th} derivative of $\ln(1+x)$.

TAYLOR SERIES

Solution(cont'd)

We derive $\ln(x+1)$ until a pattern is found.

$$f(x) = \ln(x+1),\ f^{(1)}(x) = \frac{1}{x+1},\ f^{(2)}(x) = -\frac{1}{(x+1)^2}$$

In general, $f^{(k)}(x) = \frac{(-1)^k}{(x+1)^k}$. Therefore $f^{(k)}(-2) = 1$.

TAYLOR SERIES

Solution(cont'd)

After inserting the general expression of the k^{th} derivative evaluated at -2 we obtain

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(-2)}{k!}(x+2)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(x+2)^k.$$